Identification and quantification of structural nonlinearities from measured vibration response

Dr. Alex Carrella
The Problem

![Graph showing Receptance FRF (m/N) vs Frequency [Hz] with three lines representing different forces (1 N, 5 N, 10 N) at various frequencies up to 85 Hz. The y-axis is scaled in units of $10^{-6}$ (m/N).]
The goal

taking Experimental Modal Analysis (EMA) from the current state of ideal linear world to a higher level in which nonlinearities are not ignored but, at least at a first degree of approximation, acknowledged!
- Strategy

- 4 research projects ongoing
ENGINEERING DESIGN: the HOLY GRAIL

- THEORETICAL MODELLING
- IDENTIFICATION
- SIMULATION
- EXPERIMENTAL MEASUREMENTS
- VALIDATION
- NUMERICAL ANALYSIS
My research area(s)

- Theoretical Modelling
- Identification
- Experimental Measurements
In the presence of a nonlinearity...

- Ignore it!! (Nelson principle)
- What do we do??
- I am sure we can get around it!
- Cool! Let’s dock here!
Linear system excited with stepped sine 0.1, 0.4 and 10N: FRF
Linear system (F = 0.1N): modal analysis
Linear system (F = 10N): modal analysis
Nonlinear System (QD + CS): \( F = 0.1, 0.4, 1 \text{N} \) - FRF
Nonlinear System (QD + CS): $F = 0.1N$ – modal analysis
Nonlinear System (QD + CS): $F = 10N$ – modal analysis
Nonlinear System (QD): $F = 1N$ – modal analysis

<table>
<thead>
<tr>
<th>Band</th>
<th>9.98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lines</td>
<td>204</td>
</tr>
</tbody>
</table>

**Poles**

- **Mode 1**: 10.999 Hz, 0.03 % AMPS
- **Mode 2**: 11.131 Hz, 0.81 % AMPS
- **Mode 3**: 11.186 Hz, 0.24 % AMPS

**Spectrum**

- **Freq**: 9.970 Hz
- **Damp**: 2.936 %
- **Type**: Stable
- **Tolerance**: 32
Approach to NLMT

NonLinear

Modal Testing

Spatial model
Modal model
Response model

structure

I AM SURE WE CAN GET AROUND IT!

NONLINEARITY

Receptance FRF [m/N]

Frequency [Hz]

1 N
5 N
10 N

0
0.5
1
1.5
2
2.5 x 10^-6

0 45 50 55 60 65 70 75 80 85
The major references for the study so far are:


**Time Domain Methods**: the signal analysed is acquired directly by the sensor, i.e. no need of data processing

**Frequency Domain Methods**: the system response is in the form of a FRF. This is usually preferred as it gives more visual information and physical insight than a time history.

**Modal Methods**: these attempt to extend the concept of Modal Model to a nonlinear system by extracting the modal parameters from measured data. The basic concept is the Nonlinear Normal Mode (NNM). Because is highly mathematical and theoretical further work needs to be done before it is applied to engineering problems.
Conceptually very simple. The equation of motion of a system is

\[ m\ddot{x} + f(\dot{x}, x) = y(t) \]

By knowing the excitation \( y(t) \), the response \( x(t) \) and the mass \( m \), it is possible to study the surface \( f(\dot{x}, x) = y(t) - m\ddot{x} \) which contains information on the stiffness and damping of the system.

Because the response is measured at discrete times, there is need of fitting/interpolating the surface \( f(\dot{x}, x) \). Several mathematical methods (e.g. Cebyshev polynomials) but not very physical.

\( x, \dot{x}, \ddot{x} \) and the mass need to be known. Time consuming and laborious.

\(^1\) the excitation has to contain multiple frequencies, e.g. pseudo-random, and it needs to give uniform coverage of the phase plane.
Restoring Force Surface (RFS)

\[ f_k = p_1 x^3 + p_2 x^2 + p_3 x + p_4 \]

\[ p_1 = 7 \times 10^6 \quad p_2 = -1.3391 \]

\[ p_3 = 6000.5 \quad p_4 = 0.018696 \]
• With the Harmonic Balance to a first order expansion, the response is assumed to be at the same frequency of the excitation signal.

• Inverse Method is a SDOF method based on the inversion of the Receptance FRF. However, this method is based on the assumption that the mode to be analysed is real.

• Lin (1990) extended the previous work to account also for modal complexity

• My research stems from these previous techniques and aims at defining a new approach to modal testing which does not ignore nonlinearities
CODE for Nonlinear Characterisation from Measured Response To vibration

CONCERTO
CONCERTO – the core principles

1) The mode analysed dominates the response

2) The stiffness and damping are unknown function of the amplitude of vibration, i.e. the e.o.m. is

\[ m\ddot{x} + c(X)\dot{x} + k(X)x = F_0 \sin(\omega_e t) \]

3) At a given amplitude \( (X_i) \) the system is ‘linearised’ as the damping and stiffness become coefficients

\[ m\ddot{x} + c_i\dot{x} + k_i x = F_0 \sin(\omega_e t) \]
Linearisation… (e.g. hardening cubic stiffness)

\[ K_1 < K_2 < K_3 \]
Numerical Simul. - Stepped sine excitation – 0.01Hz

\[ m\ddot{x} + c\dot{x} + k_1x + k_3x^3 = F_0 \cos(\omega_e t) \]

\( F_0 = 0.2 \text{ N} \)
\( m = 1.5 \text{ Kg} \)
\( c = 0.8 \text{ Ns/m} \)
\( k_1 = 6000 \text{ N/m} \)
\( k_3 = 7 \times 10^6 \text{ N/m}^3 \)
SDOF analysis – points around resonance

...points above and below resonance are defined at the same amplitude of vibration
At a given amplitude, correspond 2 points of the measured FRF which are:

\[ H(\omega_1) = \frac{A_r + iB_r}{\omega_r^2 - \omega_1^2 + i\eta_r \omega_r^2} = R_1 + iI_1 \]

\[ H(\omega_2) = \frac{A_r + iB_r}{\omega_r^2 - \omega_2^2 + i\eta_r \omega_r^2} = R_2 + iI_2 \]

Which can be used to find the 4 unknowns \( \omega_r, \eta_r, A_r, B_r \)

\[ \omega_r = \frac{(R_2 - R_1)(R_2 \omega_2^2 - R_1 \omega_1^2) + (I_2 - I_1)(I_2 \omega_2^2 - I_1 \omega_1^2)}{(R_2 - R_1)^2 + (I_2 - I_1)^2} \]

\[ \eta_r = \frac{- (I_2 - I_1)(R_2 \omega_2^2 - R_1 \omega_1^2) + (R_2 - R_1)(I_2 \omega_2^2 - I_1 \omega_1^2)}{\omega_r^2 (R_2 - R_1)^2 + (I_2 - I_1)^2} \]
System Identification using the Inverse Method

\[ H = \frac{1}{k - \omega^2 m + j \omega c} \Rightarrow H^{-1} = k - \omega^2 m + j \omega c \Rightarrow \begin{cases} \text{Re}(H^{-1}) = k - \omega^2 m \\ \text{Im}(H^{-1}) = \omega c \end{cases} \]
The slope of the AB line is the mass of the system (SDOF) or can be interpreted as the modal mass (MDOF).

\[ \text{Re}(H^{-1}) = k - \omega^2 m \]
Once the mass (or modal mass), $m$, has been determined, by knowing how the natural frequency and the loss factor change with amplitude ($X$), it is possible to determine the stiffness and damping as function of the amplitude, that is:

$$k(X) = \omega_r^2(X)m$$

$$c(X) = \eta_r(X)\omega_r(X)m$$
Code Validation
Numerical Simulation
Library of common nonlinearities
(from Goge’s paper)
Examples of numerical simulations

\[ m\ddot{x} + f(\dot{x}, x) = F_0 \cos(\omega_e t) \]

\[ m = 1.5 \text{Kg} \quad c = 0.8 \text{Ns/m} \]

<table>
<thead>
<tr>
<th>Nonlinearity</th>
<th>Damping Force</th>
<th>Stiffness Force</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cubic stiffness, softening</td>
<td>( c\dot{x} )</td>
<td>( k_1x + k_3x^3 )</td>
<td>( k_1 = 6000 \text{ N/m} ) ( k_3 = -4 \times 10^6 \text{ N/m}^3 )</td>
</tr>
<tr>
<td>Cubic stiffness, hardening</td>
<td>( c\dot{x} )</td>
<td>( k_1x + k_3x^3 )</td>
<td>( k_1 = 6000 \text{ N/m} ) ( k_3 = 7 \times 10^6 \text{ N/m}^3 )</td>
</tr>
<tr>
<td>Quadratic Damping</td>
<td>( c\dot{x} + c_2\dot{x}\left</td>
<td>\dot{x}\right</td>
<td>)</td>
</tr>
<tr>
<td>Pre-loaded piecewise linear stiffness</td>
<td>( c\dot{x} )</td>
<td>( k_1x, \ x &lt; b ) ( k_2x + (k_1 - k_2)b, \ b &lt; x &lt; g ) ( k_3x + (k_1 - k_2)b + (k_2 - k_3)g, \ x &gt; g )</td>
<td>( b = 0.05m ) ; ( g = 0.1m ) ( k_1 = k_3 = 9000 \text{ N/m} ) ( k_2 = 5000 \text{ N/m} )</td>
</tr>
</tbody>
</table>
CONCERTO - example

Non-linear SDOF analysis - CONCERTO
Modal mass = 1.5 kg
Quadratic Damping

![Graph showing natural frequency vs. displacement for different forces (0.2N, 0.4N, 1N). The graph indicates a constant natural frequency across the range of displacements.]
Cubic Stiffness

\[ k_{eq} = k + \frac{4}{3} k_{nl} X^2 \]
Analytical Solution – Harmonic Balance

Coulomb Damping

\[ C_{eq} = c + 4 \frac{F_f}{\pi \omega_0 X} \]

\[ C_{eq} \] vs. Displacement [m]

- **5N**
- **10N**
- **20N**

Displacement Error [%]

Extracted vs. Analytical

Damping Error [%]

- **5N**
- **10N**
- **20N**
Analysing Experimental Data with CONCERTO
Westland or NASTRAN tower

• First built in 1980 it has been extensively studied at Imperial College
• It has a softening stiffness characteristic due to the opening of the plates at the base
• Initial modal analysis at UoB agrees with the results of IC
• Acceleration measured at 12 points
Measured point FRF

![Graph showing measured point FRF with frequency vs. receptance and phase.]

- Receptance FRF $[m/N]$ vs. Frequency $[Hz]$
- Phase $[deg]$ vs. Frequency $[Hz]$

- Three lines represent different forces: 1 N, 5 N, and 10 N.
- The graphs show the response of the system at different forces and frequencies.
Measured force

![Graph showing measured force vs. frequency for different forces (1 N, 5 N, 10 N). The graph illustrates the response of the system at various frequencies, highlighting the peak at around 55 Hz for the 1 N force.]
Measured displacement spectrum

Displacement \([\text{m}]\)

Frequency \([\text{Hz}]\)

- 1 N
- 5 N
- 10 N

Displacement \([\text{m}]\) vs. Frequency \([\text{Hz}]\) for different loads (1 N, 5 N, 10 N).
CONCERTO analysis: Natural Frequency vs Displacement

- Natural frequency [Hz] vs displacement [m]
- Lines for different forces: 1 N, 5 N, 10 N
CONCERTO analysis: Damping Ratio vs Displacement

![Graph showing damping ratio vs displacement for different forces (1 N, 5 N, 10 N).](image-url)
APPLICATION TO AN
AW HELICOPTER
Measured FRFs

![Graph showing Measured FRFs with Normalised Receptance vs. Normalised Frequency, with markers for F1, F2, F3, F4, and F5.]
Constant force excitation

![Graph showing normalised force against normalised frequency with lines for F1, F2, F3, F4, and F5.](image)

- **F1**: Blue line
- **F2**: Red line
- **F3**: Black line
- **F4**: Pink line
- **F5**: Cyan line
Natural Frequency

![Graph showing natural frequencies and normalised displacement](image)
Damping ratio

![Graph showing the relationship between normalised damping ratio and normalised displacement. The graph includes lines for different cases (F1, F2, F3, F4, F5) with specific data points and trends.]
Extending CONCERTO to transmissibility measurements
Base excitation of SDOF: Transmissibility

\[ x = X e^{i\omega t} \]
\[ y = Y_0 e^{i\omega t} \]

\[ m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \]

or

\[ m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = -mY_0\omega_e^2 \sin(\omega_e t) \]

where \( z = x - y \)
Numerical simulations

\[ m\ddot{x} + f(\dot{x}, x) = y \]

where \( y = Y_0 \sin(\omega_e t) \)

The above equation is rewritten as

\[ m\ddot{z} + f(\dot{z}, z) = -\omega_e^2 Y_0 \sin(\omega_e t) \]

Solved with MATLAB built-in 4\(^{th}\) order Runge-Kutta ODE45.

The absolute displacement is computed as \( x = z + y \).

The Transmissibility is then computed as the ratio between the Fourier coefficients (at the excitation frequency) of the output and input, that is

\[ T(\omega_e) = \frac{X}{Y_0} = \frac{\sum xe^{-i\omega_e t} dt}{\sum ye^{-i\omega_e t} dt} \]
Examples of numerical simulations

\[ m\ddot{x} + f(\dot{x}, x) = Y_0 \sin(\omega_0 t) \]

\[ m = 1.5\text{Kg} \quad c = 0.8\text{Ns/m} \]

<table>
<thead>
<tr>
<th>Nonlinearity</th>
<th>Damping Force</th>
<th>Stiffness Force</th>
<th>Values</th>
</tr>
</thead>
</table>
| Mixed cubic stiffness + quadratic damping | \( c\dot{x} + c_2\dot{x}|\dot{x}| \) | \( k_1x + k_3x^3 \) | \( k_1 = 6000\text{ N/m} \)
|                                    |                         |                     | \( k_3 = -4\times10^6\text{ N/m}^3 \) |
| Cubic stiffness, hardening          | \( c\dot{x} \)         | \( k_1x + k_3x^3 \) | \( k_1 = 6000\text{ N/m} \)
|                                    |                         |                     | \( k_3 = 7\times10^6\text{ N/m}^3 \) |
| Quadratic Damping                  | \( c\dot{x} + c_2\dot{x}|\dot{x}| \) | \( k_1x \)       | \( k_1 = 6000\text{ N/m} \)
|                                    |                         |                     | \( c_2 = 8\text{ Ns/m}^2 \) |
| Coulomb Damping                    | \( c\dot{x} + \mu N_f \text{sign}(\dot{x}) \) | \( k_1x \)       | \( \mu N_f = 0.85\text{m} \)
|                                    |                         |                     | \( k_1 = 6000\text{ N/m} \) |
Transmissibility: quadratic damping

- $Y_0 = 0.15\text{mm}$
- $Y_0 = 0.3\text{mm}$
- $Y_0 = 0.5\text{mm}$

Frequency [Hz] vs. Transmissibility
CONCERTO results: Coulomb damping

- Natural Frequency [Hz]
- Damping Ratio [%]
- Displacement [m]

- Y₀ = 1mm
- Y₀ = 2.5mm
- Y₀ = 5mm
FRF and Transmissibility analysis

Quadratic Damping
FRF and Transmissibility analysis

Coulomb Damping
FRF and Transmissibility analysis

Mix cubic stiffness and quadratic damping
How/Why does it work?
Base excitation of SDOF: Transmissibility

For a SDOF system and harmonic excitation of the base the Transmissibility is defined as

\[ m\dddot{x} + k(1 + i\eta)x = k(1 + i\eta)y \]

\[ T = \frac{X}{Y_0} = \frac{\omega_0^2 (1 + i\eta)}{\omega_0^2 - \omega^2 + i\eta\omega_0^2} \]
In both cases the numerators are equal (at that displacement) and cancel-out but denominator is the same, hence the natural frequency and damping loss factor can be extracted using the algorithm of CONCERTO in both cases!
Using CONCERTO with Transmissibility

\[ T_1 = \frac{X}{Y_0} = \frac{\omega_0^2(1 + i\eta)}{\omega_0^2 - \omega_1^2 + i\eta\omega_0^2} = R_1 + iI_1 \]

\[ T_2 = \frac{X}{Y_0} = \frac{\omega_0^2(1 + i\eta)}{\omega_0^2 - \omega_2^2 + i\eta\omega_0^2} = R_2 + iI_2 \]

\[ \omega_0^2 = \frac{(R_2 - R_1)(R_2 \omega_2^2 - R_1 \omega_1^2) + (I_2 - I_1)(I_2 \omega_2^2 - I_1 \omega_1^2)}{(R_2 - R_1)^2 + (I_2 - I_1)^2} \]

\[ \eta = \frac{-(I_2 - I_1)(R_2 \omega_2^2 - R_1 \omega_1^2) + (R_2 - R_1)(I_2 \omega_2^2 - I_1 \omega_1^2)}{\omega_r^2 (R_2 - R_1)^2 + (I_2 - I_1)^2} \]
AVM Testing
(burst) RANDOM excitation

Isolation Region

Transmissibility vs. Frequency [Hz]
## (burst) RANDOM excitation - results

<table>
<thead>
<tr>
<th>Mass</th>
<th>Nat Freq [Hz]</th>
<th>Maximum Transmissibility</th>
<th>Damping ratio [%]</th>
<th>Isolation region</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>24.13</td>
<td>4.4</td>
<td>11.36</td>
<td>34.12</td>
</tr>
<tr>
<td>M2</td>
<td>24</td>
<td>5.19</td>
<td>9.63</td>
<td>33.94</td>
</tr>
<tr>
<td>M3</td>
<td>15.88</td>
<td>4.55</td>
<td>10.99</td>
<td>22.46</td>
</tr>
<tr>
<td>M4</td>
<td>16</td>
<td>3.85</td>
<td>12.99</td>
<td>22.63</td>
</tr>
<tr>
<td>M5</td>
<td>11.13</td>
<td>3.33</td>
<td>15.02</td>
<td>15.74</td>
</tr>
<tr>
<td>M6</td>
<td>10.63</td>
<td>3.36</td>
<td>14.88</td>
<td>15.03</td>
</tr>
</tbody>
</table>
Stepped-Sine excitation - input

Base Acceleration [g]

Frequency

0.25g
0.5g
0.75g
Stepped-Sine excitation

![Graph showing transmissibility vs frequency with lines for 0.25g, 0.5g, and 0.75g.]
Stepped-Sine excitation: linearity plot
Stepped-Sine excitation: linearity plot

Damping Ratio [%] vs. Base acceleration [g]

- M1
- M2
- M3
- M4
- M5
- M6
Using CONCERTO with Transmissibility

![Graph showing the relationship between natural frequency and mount displacement. The graph plots natural frequency in Hz on the y-axis and mount displacement in mm on the x-axis. There are two curves, one in blue and one in cyan, demonstrating the change in natural frequency as the mount displacement increases.]
Using CONCERTO with Transmissibility

![Graph showing the relationship between natural frequency (Hz) and relative displacement (mm) for different accelerations (0.25g, 0.5g, 0.75g) and types of excitation (Burst-random, Rap)].

- 0.25g
- 0.5g
- 0.75g
- Burst-random
- Rap
Using CONCERTO with Transmissibility

![Graph showing natural frequency vs. displacement with different mass loads (0.25 g, 0.5 g, 0.75 g).]
Using CONCERTO with Transmissibility

![Graph showing stiffness vs. displacement for different masses (0.25 g, 0.5 g, 0.75 g). The graph plots stiffness in N/m on the y-axis and displacement in m on the x-axis. The graph shows a downward trend as displacement increases for all masses.]
Using CONCERTO: a nonlinear model

\[
\left[ -\omega^2 m + j\omega c + k(z) \right] Z = \omega^2 mY
\]
Work in progress
Cantilever beam (linear)
'Measured' FRF

Response/measurement points

\[ F = F_0 \cos(\omega t) \]
CONCERTO: natural frequency
CONCERTO: damping ratio

![Graph showing damping ratio vs. displacement. The graph indicates the damping ratio in percentages and displacement in meters, illustrating a non-linear relationship between the two. The horizontal axis represents displacement in meters, while the vertical axis represents damping ratio. The graph has a dashed line that starts from a low damping ratio at a certain displacement, rises to a peak, and then drops significantly before leveling off.]
CONCERTO: mode shape
Use CONCERTO to analyse the FRF of the nonlinear beam:

- Modal parameters: global or local (?)
- Mode shapes (?)
- Localisation (?)
Thank You

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Questions
Pleasure meeting you
1. Dr Alex Carrella

- Degree in Aeronautical Engineering, University of Naples “Federico II”, Italy (project/placement aeroelasticity)
- PhD in Structural Dynamics (“Passive Vibration Isolator with High-Static-Low-Dynamic-Stiffness”) ISVR, University of Southampton, UK
- Dynamic analysis of bi-stable plates: The Morphing Aircraft, postdoc, University of Bristol, UK
- AgustaWestland UTC, postdoc, University of Bristol, UK
- Lecturer in Structural Dynamics, University of Bristol, UK
2. University of Bristol
2. University of Bristol
2. University of Bristol

6 Faculties
13,000 undergrad

3000 research students
5800 staff
(2000 research & teaching)

www.bristol.ac.uk/university
Faculty of Engineering

Departments
- Aerospace Engineering
- Civil Engineering
- Computer Science
- Electrical and Electronic Engineering
- Engineering Mathematics
- Mechanical Engineering

Interdisciplinary Degree Programmes
- Computer Science and Electronics
- Engineering Design
- Masters and doctorate level programmes in Systems

University Research Centres
- Advanced Composites Centre for Innovation and Science (ACCIS)
- Bristol Laboratory for Advanced Dynamics Engineering (BLADE)
- Centre for Communications Research (CCR)
- Earthquake Engineering (EERC)
- Safety Systems Research Centre (SSRC)
- Water and Environmental Management Research Centre (WEMRC)
3. Bristol Laboratory for Advanced Dynamics Engineering

Earthquake lab

Dynamic lab
Bristol Laboratory for Advanced Dynamics

Engineering

Earthquake lab

Dynamic lab
3. BLADE: the next decade

Centre for High Performance Testing

Prof. DJ Ewins

- Modal testing for Model Validation
- Residual stress evaluation
- NDT methods
- Endurance testing of composite materials
- Substructure (Hybrid) testing
- ‘Smart’ Testing

Institute for Smart Technologies

Prof. D Inman

- Multifunctional materials
- Adaptive structures
- Smart machines
- Structural health monitoring
- Energy harvesting
- Embedded Intelligent devices

....
4. AgustaWestland UTC
What is NLMT?

Why there is need of NLMT?

How do WE do NLMT?
Time Domain Methods for NLMT

- **Restoring Force Surface (RFS)**: first presented by Masri and Caughey in 1979 (aka Force State Mapping)

- **Nonlinear Auto-Regressive Moving Average with eXogeneous input (NARMAX)**: initially developed for Control Engineering problems it was used in structural dynamics first in the 70’s (ARMA models), then extended to nonlinear systems (Billing, 1985)

- **Hilbert Transform**: pioneered by Tomlinson in the late 70’s was brought to attention by Michael Feldman who proposed FREEVIB and FORCEVIB methods
Hilbert Transform (HT)

• A ‘classic’ approach for using the HT in structural dynamics analysis of nonlinear system is in the Frequency Domain.

• However, there is exist a fundamental problem in using the HT on FRF data which have measured over limited range of frequencies. Methods that can correct this ‘fault’ have been proposed. However, a newer technique has been developed which exploits the properties of the HT in the Time Domain.

• The method is attributed to Michael Feldman who first developed it in 1985 and then coded FREEVIB and FORCEVIB in the mid-90’s
Consider the nonlinear system described by

\[ \ddot{y} + h(\dot{y})\dot{y} + \omega_0^2(y)y = 0 \]

the response (motion) is \( y(t) \) (NOTE that this is a measurable quantity!)

From the response is possible to build the \textit{analytical signal} defined as

\[ Y(t) = y(t) + j\tilde{y}(t) = A(t)e^{j\psi(t)} \]

Where \( \tilde{y}(t) \) is the Hilbert Transform of the signal \( y(t) \),
\( A(t) \) is the envelop of the signal amplitude and
\( \psi(t) \) is the instantaneous phase

Finally, after some algebra, the instantaneous natural frequency and damping can be calculated
Hilbert Transform (HT): FORCEVIB

- The method is based on the theory shown earlier;
- The mass needs to be also calculated
- Excitation signal has to contain a range of frequency, ideal for sine-sweep.
- One basic assumption of this method is that damping and restoring force are symmetric
- Another shortfall of this technique is that the response signal needs to contain only one mode, i.e. MODF systems cannot be (easily) analysed. An extension of the method is referred to as **Huang-Hilbert transform**. Feldman has also proposed a signal decomposition algorithm.
Hilbert Transform (HT): FORCEVIB

Graphs and plots showing data related to force, displacement, spring force, damping force, velocity, and frequency.
Frequency Domain Methods for NLMT

• **Higher Order Frequency Response Function (HOFRF):** functional series such as Volterra or Wiener extend the concept of Impulse Response Function to nonlinear systems.

• **Reverse Path analysis:** developed in the late 80’s and its output are nonlinear coefficients and the underlying linear model. It is based on point FRF measurements. A later version is the **Conditioned Reverse Path (CRP).**

• **Harmonic Balance – Inverse Method**
Transmissibility : Coulomb damping

![Graph showing transmissibility with different Y_0 values (1mm, 2.5mm, and 5mm) against frequency (Hz).]
CONCERTO results: cubic stiffness

Natural Frequency [Hz]

Damping Ratio [%]

Displacement [m]
CONCERTO results: quadratic damping

- **Natural Frequency [Hz]**
- **Damping Ratio [%]**
- **Displacement [m]**

Graphs showing the relationship between natural frequency, damping ratio, and displacement for different initial displacements ($Y_0$).

- $Y_0 = 0.15\text{mm}$
- $Y_0 = 0.3\text{mm}$
- $Y_0 = 0.5\text{mm}$
CONCERTO results: CS + QD

- **Natural Frequency [Hz]**
  - Blue line: $Y_0 = 0.15\text{mm}$
  - Red line: $Y_0 = 0.3\text{mm}$
  - Black line: $Y_0 = 0.5\text{mm}$

- **Displacement [m]**
- **Damping Ratio [%]**

Displacement vs. Damping Ratio and Natural Frequency vs. Displacement graphs are shown.