Instructions

• You have 24 hours to complete this exam.

• Calculators are permitted.

• Show your working clearly. Answers without working will not be given full credit.

• The marks for each question part are shown in brackets.
1. Let \( w_1 = 3 + 4i \) and \( w_2 = -1 + 2i \).
   (a) Find the following, giving your answers in the form \( a + bi \).
   i. \( w_1^* w_2 \) \[2\]
   ii. \( \frac{w_1 - 3w_2}{w_2} \) \[2\]
   (b) Write down a quadratic equation with real coefficients that has \( w_1 \) as a root. \[2\]

2. The graph of the function \( y = x^2 - 2x - 3 \) is sketched below. Find the area of the shaded region. \[4\]

3. Given that \( \mathbf{i} \) and \( \mathbf{j} \) represent two perpendicular unit vectors,
   (a) Find the magnitude of the vector \( -8\mathbf{i} + 15\mathbf{j} \) \[2\]
   (b) Find the angle \( \theta \) between the vector \( -8\mathbf{i} + 15\mathbf{j} \) and the vector \( \mathbf{j} \), giving your answer in degrees to 2 decimal places. \[4\]

4. In 2005 there were 20 puffins on Lundy island. Counting \( t = 0 \) as the year 2005, after \( t \) years the number of puffins on the island, \( P \), is modelled by the equation
   \[
   \frac{dP}{dt} = 0.2P
   \]
   (a) Find an equation for \( P \) in terms of \( t \). \[5\]
   (b) Hence estimate, to the nearest hundred, how many puffins there will be on Lundy island in the year 2030. \[2\]
   (c) Is this model suitable for predicting how many puffins will be on the island in the year 2100? Give a reason for your answer. \[1\]

5. Let \( w = 5e^{3\pi i/4} \) and \( z = 4e^{\pi i/2} \).
   (a) State the value of \( |wz| \). \[1\]
   (b) State the value of \( \arg \left( \frac{w}{z} \right) \). \[1\]
   The complex number \( v \) satisfies \( v^2 = z \), with \( z \) as above.
   (c) Write down the possible values of \( v \) in the form \( re^{i\theta} \) with \( r > 0 \) and \( -\pi < \theta \leq \pi \). \[2\]
6. The matrices $A$ and $B$ satisfy the equation 
\[ AB = I - 2A \]
where $I$ is the identity matrix and 
\[ B = \begin{pmatrix} 5 & -4 \\ -3 & 0 \end{pmatrix}. \]
Find $A$. \[3\]

7. It is given that $x$ and $y$ are related by the ordinary differential equation 
\[ \frac{dy}{dx} = 8 - \frac{3y}{1+x} \]
and that when $x = 0$, $y = -14$. 
Find the value of $y$ when $x = 1$. \[6\]

8. Find a vector that is perpendicular to both 
\[ \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} \] 
and 
\[ \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}. \] \[3\]

9. A complex number $z$ has modulus 1 and argument $\theta$.
   (a) Show that 
   \[ z^n + \frac{1}{z^n} = 2 \cos(n\theta) \]
   for any integer $n$. \[3\]
   (b) Hence show that 
   \[ \cos^5 \theta = \frac{1}{16} (\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta) \] \[6\]

10. (a) Find 
\[ \int 9x^5 \sqrt{x^3 + 2} \, dx \] \[5\]
   (b) Find 
\[ \int_{\ln 2}^{\ln 4} 2ue^{2u} \, du, \] giving your answer in the form $a \ln 2 + b$, where $a$ and $b$ are constants to be found. \[4\]

11. Consider the simultaneous equations 
\[ \begin{align*} 
3x + 4y &= 6 \\
-x + 4y &= 19 
\end{align*} \]
   (a) Write these equations as a matrix equation $Ax = b$. \[2\]
   (b) Find the determinant of the matrix \[ \begin{pmatrix} 6 & 4 \\ 19 & 4 \end{pmatrix}. \] \[2\]
   (c) Use Cramer’s rule to solve the simultaneous equations. You must show detailed reasoning. \[3\]
12. The sketch below shows the curves \( y = A - 2\sqrt{x} \) and \( y = x^2 - B \) for \( x \) in the range \( 0 \leq x \leq a \), where \( A, B, \) and \( a \) are all positive constants. The sketch is not drawn to scale. Both curves meet the \( x \)-axis at \( x = a \).

(a) Find the area of the finite region bounded by the \( y \)-axis and the two curves, giving your answer in terms of \( A \).

The area of this region is 504.

(b) Find the value of \( A \).

(c) Hence find the values of \( B \) and \( a \).

13. The points \( A(4, 5, -1) \), \( B(4, -5, 8) \), \( C(6, 1, -1) \), and \( D(6, 11, -10) \) are the four corners of a parallelogram. Find the area of the parallelogram \( ABCD \).

14. (a) Sketch an Argand diagram showing the locus of points satisfying the equation

\[
|z + 2| = 2.
\]

(b) Given that there is a unique complex number \( w \) that satisfies both

\[
|w + 2| = 2
\]

and

\[
\text{arg}(w - k) = \frac{3\pi}{4},
\]

where \( k \) is a positive real number,

i. Find the value of \( k \).

ii. Express \( w \) in the form \( r (\cos \theta + i \sin \theta) \), giving \( r \) and \( \theta \) to 2 significant figures.
15. A particle moves around a fixed point $O$ with damped harmonic motion. The displacement in metres of the particle from $O$ at time $t$ seconds is denoted by $x$, where $x$ satisfies the equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$$

(a) Find the general solution to this differential equation. [3]

When $t = 0$ the particle is a distance of 15 m from $O$ and is moving with velocity $5 \text{ ms}^{-1}$.

(b) Find an expression for $x$ in terms of $t$. [3]

(c) Briefly explain what happens to the particle as $t$ increases. [1]

A second particle has displacement $u$ from $O$ at time $t$ given by the equation

$$\frac{d^2u}{dt^2} + 6\frac{du}{dt} + 9u = e^{-3t}$$

(d) Find the general solution to this equation, expressing $u$ in terms of $t$. [6]

END OF EXAM.