The Structure of Classical Mechanics

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Abstract

How do we learn about the nature of the world from the mathematical formulation of a physical theory? One rule we follow, familiar from spacetime theorizing: posit the least amount of spacetime structure required by the fundamental dynamics. I think we should extend this rule beyond spacetime structure. We should extend the rule to statespace structure. Using this rule, I argue that a classical world has a surprisingly spare amount of structure. I draw the moral that we should be realists about a theory's statespace structure.

If something is in me that can be called religious then it is the unbounded admiration for the structure of the world so far as science can reveal it.—*Einstein* (from a letter written in 1954)

1. Introduction

We look to our physical theories to tell us what the world is like. We look to our fundamental physical theories to tell us what the world is fundamentally like. If anything can tell us about the world at its most basic level—the nature of space and time, the ultimate constituents of matter, what those constituents are like—fundamental physics can. A fundamental physical theory tells us what the world *must* be like, at bottom, in order for the theory to correctly describe the world, in order for it to be true of the world.

It is not obvious, though, how our physical theories, formulated in mathematical language, manage to tell us about the world. How do we extract from a mathematical formalism various features of the physical world? How does the abstract structure of a physical theory depict the concrete world of our experience?

We need some rules or guidelines. Here is one rule we follow, familiar from spacetime theorizing: infer the least amount of spacetime structure presupposed by the world's fundamental dynamics. This rule tells us that the spacetime of a special relativistic world lacks an absolute velocity structure, for example. The dynamical laws of the theory are invariant under transformations that alter a system's velocity; they "say the same thing" regardless of a choice of velocity. According to special relativity, then, there is no such thing *the* velocity of an object. There is only the velocity relative to a choice of inertial frame, and no inertial frame is preferred. Hence, we infer, the spacetime of the theory lacks the structure to distinguish, in a frame-independent way, timelike trajectories that are orthogonal to the simultaneity planes from those that are not.

The rule won't yield conclusive inferences. There could still be a preferred rest frame, for example. But it is a reasonable guide even so. It is a rule similar in temperament to Occam's razor, but is less weighty than that. It is simply a guiding methodological principle—a principle which, it happens, has been successful in our physical theorizing.

I think we should extend this rule beyond spacetime structure. We should extend the rule to statespace structure. The structure of a theory's statespace is also given by invariant quantities under allowable transformations of the dynamical laws. This suggests that, just as with a theory's spacetime, a theory's statespace corresponds to genuine structure in the world. I will argue for this by comparing two different versions of classical mechanics, Lagrangian and Newtonian mechanics. A comparison of these theories' statespaces—in just the way that we compare the structures of different spacetimes—will reveal that a classical world has a surprisingly spare amount of structure; much less than what we ordinarily think. Although I won't give explicit argument for applying our rule to statespace structure, I do hope to show, by looking closely at these two theories, that we should take the structure of a theory's statespace seriously—as seriously as we take the structure of a theory's spacetime. More: that we should be realists about statespace structure. We should think that this structure *exists*.

2. Inferring structure

First, a bit more on the above rule for inferring the structure of the world from the mathematical formulation of a physical theory.

There are two parts to the rule, equally important. The first is the claim that mathematical features of the laws—in particular, their symmetry, or *invariance*, under certain transformations[1]—represent structural features of the world. The second is the injunction to infer the least such structure as indicated by these features.

According to the first part of the rule, invariances or non-invariances in the dynamical laws indicate structural features of the world, where invariances indicate symmetric structure and non-invariances asymmetric structure. The laws are invariant when they remain the same after undergoing a transformation. If they do remain the same, we infer that the structure of the world underlying the transformation is symmetric. If they do not, we infer that the underlying structure is asymmetric, giving rise to the laws' non-invariance. If the laws of special relativity were not invariant under Lorentz transformations, for example, we would infer an absolute velocity or preferred-frame structure to the world.

The reason that symmetries are so important is they indicate which features result from mere arbitrary choices in description, not distinctions in reality. They indicate which features don't, in the end, *matter* to a theory. In other words, (a-)symmetries or (non-)invariances in the laws reveal the underlying *structure* of the world, according to a theory: the intrinsic, absolute, objective nature of the world. Thus, in special relativity, we can choose a Lorentz frame for the purpose of describing a system, but that choice is arbitrary. The theory itself doesn't distinguish among different frames: the laws remain the same under different choices of frame. We conclude that frame-relative quantities, like velocity, are not genuine features of the world in this theory. The spacetime interval, on the other hand, is an absolute, structural feature of the world, according to the theory. All allowable reference frames have this structure.

The second part of our rule says to infer the least structure indicated by the laws. And although comparing abstract structures may seem an

[1] A transformation is just a function or mapping, in this case from the laws to themselves, or solutions to the laws to solutions.

imprecise task, there is one respect in which it is not: symmetries are the mark of a lesser structure. When two structures, otherwise comparable, differ with respect to a symmetry, the one with the symmetry has the lesser structure. A corollary of our rule then says: don't attribute to the spacetime of a world any asymmetries not had by the dynamical laws. Conversely, any symmetries in the dynamical laws should be reflected in spacetime symmetries of the world.[2]

This stems from the way we compare abstract structures in mathematics. We compare the structures of different mathematical spaces by looking at how many "levels" are needed to define the space, starting with the most basic level, such as sets of points.[3] This is what allows us to say that a topological space has "more structure" than a bare set—a collection of objects with "no further structure"—and that a metric space has "more structure" than a topological space. A topology adds structure to a bare set of points (a topology says which subsets are open and which are closed); and a metric adds structure to a topological space (a metric allows us to calculate distances between points). And in general, an additional level of structure is needed for a space to possess an asymmetry. Intuitively, a set is just a structureless collection of objects; some further mathematical object is needed to pick out a preferred element in the set.

A bit more precisely, given two mathematical spaces with the same amount of structure, there will be a structure-preserving mapping between them. If two spaces share a level of structure but differ with respect to an asymmetry, the asymmetric structure will have "more left over" when we do this mapping.[4] Another way of putting this: the group of symmetries respected by the symmetric structure will be larger than the group of symmetries respected by the asymmetric structure; the latter will be a proper subset (its symmetries will form a proper subgroup) of the former.

In mathematics, a comparison of structure indicates, for example, that a Euclidean plane with a preferred spatial origin has more structure

[2] Earman (1989, 46) considers this a condition of adequacy for dynamical theories.

[3] This is not to take a stand the proper foundational level for mathematics. It suffices that there be a clear hierarchy, starting from a chosen level (sets, categories). I'll assume sets of points, the usual starting point for physical theories.

[4] Cf. Sklar (1974, 48-49).

than a Euclidean plane without one; the two planes share some common structure, but the former has an additional, preferred-point structure. Similarly in physics. Comparison of structure indicates that a spacetime with a notion of absolute velocity has more structure than a spacetime without one. That is why, since the laws of special relativity can be formulated without assuming an absolute velocity, we infer—in accord with our rule—that the spacetime of the theory doesn't have this extra structure.

Although the rule is most familiar for relativistic theories, we can put it to work in classical theories, too. The dynamical laws of classical mechanics are invariant under transformations that alter a choice of temporal or spatial origin, for example. As in the relativistic case, the invariances here are mathematical properties of the laws that we can check by means of a simple mathematical procedure: choose a different temporal or spatial origin, and the laws have the same mathematical form. Since a spacetime with a preferred origin has more structure than a spacetime without one, and since we can formulate the classical laws without implicitly referring to such a structure, we infer—in accord with our rule—that the spacetime of the theory doesn't have this extra structure; that it is symmetric among the temporal and spatial locations.

More broadly, the rule tells us that the proper spacetime structure for the classical dynamics is Galilean (or neo-Newtonian), not Aristotelian (Newtonian). Comparison of structure indicates that the latter spacetime has more structure. Galilean spacetime has (affine) structure picking out the inertial trajectories, but it lacks the structure to identify cross-temporal spatial locations. Aristotelian spacetime has both kinds of structure. Thus, we need to add structure to a Galilean spacetime to get the full structure of a Newtonian spacetime. And since the Newtonian laws don't require that extra structure, we infer—in accord with our rule—that the proper spacetime structure of the theory is Galilean rather than Aristotelian.[5]

So far, I've only mentioned (relatively uncontroversial) uses of our rule to infer a theory's spacetime structure. A more general version of the

[5]Cf. Table 2.1 in Earman (1989, 36). See Stein (1970); Arnol'd (1989, 1.2); Szekeres (2004, 54-55); DiSalle (2006) for more detail on this structure and why it is the appropriate spacetime for Newtonian dynamics.

rule would not be limited to the structure of spacetime. A general version would say something like: infer that the world has the least amount of structure needed to support the fundamental dynamical laws; "support" in the sense that the laws can't be formulated without implicitly referring to that structure. Infer the least amount of structure to the world—that is, structure *in general*, not just spacetime structure—that the mathematical formulation of its fundamental physics can get away with.

The idea behind the general version of our rule is simple. If the fundamental laws can't be formulated without implicit reference to some structure, then that is reason to think the structure represents genuine features of the world. If they can be so formulated, then that is reason to think it mere excess, superfluous structure: an artifact of the mathematical formalism, not genuine structure in the world. And the way to find out about this structure is to look for invariances in the laws. Note that when investigating the structure of the world, we should look for invariances in its *fundamental* laws. Non-fundamental laws may be asymmetric in a way that does not indicate any underlying asymmetric structure. (Think of thermodynamics and the asymmetry of time.)

Let's now look at Lagrangian and Newtonian mechanics and see what this rule may be able to tell us about the fundamental nature of a classical world.

3. Newtonian mechanics: a quick and dirty review

The Newtonian formulation is the one we're most familiar with. I run through a few of the basics here, highlighting the features that will be important to us later.

In Newtonian mechanics,[6] we need two sets of coordinates to completely characterize the fundamental state of a system at a time: the positions and momenta (or velocities) of all the particles. (This is in addition to the particles' intrinsic features, such as mass and charge.[7]) For a system that is moving freely in a three-dimensional physical space, the

[6] This version may not be Newton's own. For discussion, see Stein (1970, 1977a,b); DiSalle (2006).

[7] I leave aside the question of whether velocity is a property of an instantaneous state: Albert (2000); Arntzenius (2000); Earman (2002); Smith (2003); Arntzenius (2003).

position and momentum coordinates will each have three components, one along each spatial dimension.

These coordinates characterize the possible fundamental states of a Newtonian system. They also characterize the statespace, the mathematical space in which we represent a system's possible states. In Newtonian mechanics, a system with *n* particles will have a statespace of 6*n* dimensions: one for each of the three position and momentum coordinates for each particle. Each dimension corresponds to one of the 6*n* coordinates that specify the state, and each point picks out a possible state.

A curve or trajectory through the statespace represents a possible history for the system. The dynamical laws of a theory say which of these curves correspond to possible histories. In Newtonian mechanics, the second law governs this evolution:

$$\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) = m\frac{d^2\mathbf{x}}{dt^2} = m\ddot{\mathbf{x}} = m\mathbf{a}.^8 \tag{1}$$

This is a vector equation. There is one such equation for each particle in each component direction. We can group these component equations into one master version of the law, which says how the point representing the entire state of the system moves through the statespace over time.

Given the initial positions and momenta of all the particles in a system, and given the total forces on the system (represented by a vector function on the statespace), we can twice integrate Newton's law to get a unique solution. (The laws are deterministic.[9]) A solution represents the history of the system, for a given initial state and subject to these forces. This will have the mathematical form of a vector function $\mathbf{x}(t)$ for initial $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$. The reason we need two different sets of initial information is that the equation of motion is of second order. Two sets of initial values are needed to solve for the two constants of integration.

[8] Or $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, where $\mathbf{p} = m\dot{\mathbf{x}} = m\mathbf{v}$. This version explicitly allows for the mass to change with time, a generalization we won't need here.

[9] Another way of stating Newton's law is that there is a function $\mathbf{F} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ such that $\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$. By the theorem of existence and uniqueness of solutions to ordinary differential equations, the function $\mathbf{F}$ and the initial conditions $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$ uniquely determine the motion, under certain smoothness conditions (Arnol'd, 1989; Arnol'd et al., 1997, ch. 1). I set aside the intriguing cases of indeterminism, when those conditions are not met: Earman (1986); Malament (2006); Norton (2006).

Newton's second law is *the* dynamical law of the theory. This law determines the motions of all the particles, in any system, in any situation. The forces acting on a system will depend on the types of particles involved; and to calculate the forces, we may need additional rules, such as Coulomb's law or Newton's law of gravitation. But the one dynamical law determines the system's behavior once we have those forces. As Albert (2000, 1-2) puts it, a Newtonian world can be summed up by three fundamental principles: the dynamical law $\mathbf{F} = m\mathbf{a}$; the claim that everything is made up of particles with certain intrinsic features; and that all forces are functions of inter-particle distances and those intrinsic features.

A few simple examples help to get a feel for the theory and the way it describes classical systems. Take a single particle[10] of mass m constrained to move along a finite segment of a straight line, such as a bead on a straight (frictionless) wire. This is a system with one degree of freedom. Let the distance along the line (the linear displacement away from a chosen origin) be x, and let the force acting on the particle, also along the line, be a function only of x.[11] The equation of motion is then

$$F(x) = m\ddot{x}. \tag{2}$$

If the particle undergoes simple harmonic motion, for example, for which $F(x) = -kx$, k a constant, the equation of motion will be

$$-kx = m\ddot{x}.^{12} \tag{3}$$

If the particle falls freely from rest, then the force is that due to gravity, mg, and the equation of motion is

$$g = \ddot{x}. \tag{4}$$

Integrate the equation of motion once to get $\dot{x}(t)$ and again to get $x(t)$, the position and velocity of the particle as functions of time.[13] Plugging

[10] I stick to single particles. The extension to many-particle systems is straightforward.

[11] In general, F can be a function also of t and $\dot{x}$. I stick to the simpler cases here.

[12] This is often rewritten in the form $\ddot{x} + \omega^2 x = 0,\ \omega = k/m$.

[13] For the particle in free fall, choose the origin to be its starting point and integrate to get the solution $x(t) = \frac{1}{2}gt^2$. For simple harmonic motion, define $\omega^2 = \frac{k}{m}$ and assume small angular displacements. This yields the equation of motion $\ddot{x} + \omega^2 x = 0$. The solution has the general form $x(t) = A\cos\omega t + B\sin\omega t$, A and B constants.

in the initial state, given by $x(t_0)$ and $\dot{x}(t_0)$, yields a unique solution; here, a vector function in one dimension.

(Alternatively, we can find solutions to these examples using conservation of energy. Since the force on the particle is conservative, the force function will be the negative derivative of a potential energy function: $F(x) = -\nabla V(x)$. Integrating $F(x) = m\ddot{x}$ once, we get the equation of motion in the form $\frac{1}{2}m\dot{x}^2 + V(x) = E$, where E is the total energy. Solve this equation for $\frac{1}{\dot{x}}$ and integrate to get our solution. We'll return to this.)

In the interest of foreshadowing, consider what seems to be a similar example. Take a single particle of mass m constrained to move along an arclength of a circle, again with a force that is only a function of position; in this case, the position along the arclength. This, too, is a system with one degree of freedom, and it seems as though we should be able to find the solution in the same way. Yet it turns out to be a bit more difficult than that. The reason is simple: Newton's law privileges certain kinds of coordinate systems, the linear, Cartesian, rectangular ones. When the path of a system or the net force acting on it is a function of an angular displacement, Newton's law will take a different mathematical form.

To illustrate, take a vertical plane pendulum, shown in Figure 1. (Assume the usual idealizations: frictionless light string, point mass, negligible air resistance.) The force affecting the motion is the force due to

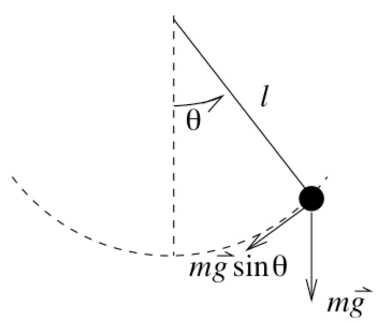

Figure 1: Plane pendulum (image by MIT OCW)

gravity. (The tension in the string, perpendicular to the motion of the pendulum bob, does no work.) In order to apply Newton's second law and solve for the motion of the pendulum bob, we choose a rectangular coordinate system and resolve the force into its components. Thus, choose the y-direction to be the radial direction (the direction along the string) and the x-direction to be the tangential direction (tangential to the path of the bob). Resolve the total force on the bob into its components along these directions. The relevant component of the force—the component that is in the direction of the acceleration of the bob—is $mg \sin\theta$, where θ is the angle the string makes with respect to the vertical; see Figure 1.[14] The arclength s, which measures the distance traveled by the bob along the curved path swept out by the pendulum, is given by $s = l\theta$, where l is the length of the string.[15] Calculate the derivatives $\dot{s} = l\dot{\theta}$ and $\ddot{s} = l\ddot{\theta}$ and plug into Newton's law. Since $F(\theta) = -mg\sin\theta$, the equation of motion is $F = ml\ddot{\theta} = -mg\sin\theta$, or

$$-g\sin\theta = l\ddot{\theta}.^{16} \tag{5}$$

Plug in the initial state, given by initial values $\theta(t_0)$ and $\dot{\theta}(t_0)$, and integrate twice to get the solution.[17]

Notice that the equation of motion for the pendulum does not take the simple form it did for the particle on a straight line. The equation of motion for the pendulum (equation 5) is this:

$$-g\sin\theta = l\ddot{\theta}.$$

The equation for the particle on a straight line (equation 2) is this:

$$F(x) = m\ddot{x},$$

where $F(x)$ is a function such as $-kx$ or mg. Unlike the particle moving in a straight line, the equation of motion for the pendulum is not just a

[14]Assume that the pendulum moves through a uniform gravitational field.

[15]Really we need to integrate a differential version of this quantity along the path.

[16]This is often rewritten as $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$, or $\ddot{\theta} + \omega^2\sin\theta = 0$, where $\omega = \sqrt{\frac{g}{l}}$.

[17]It's not quite so simple as that, since this is a nonlinear equation. But if we assume small θ, so that $\sin\theta \approx \theta$ (for the oscillation near the bottom of the swing), then the pendulum approximates simple harmonic motion. This gives $\ddot{\theta} + \omega^2 = 0$, a linear differential equation with solution as in note 13. This is only accurate in small angles because the angular acceleration is proportional to the sine of the position. When θ is small, though, $\sin\theta \approx \theta$, and the angular acceleration is directly proportional to θ, as in simple harmonic motion. More on this soon.

function of a position variable (here, θ) and its time derivatives; it is also a function of the sine of that variable.

Why should the equation of motion have a different form? Two reasons. First, Newton's equation is a vector equation, an equation that relates vector quantities; and vector equations hold component-wise. There's a transformation rule that holds for all vectors (this rule can be taken to define vectors), relating the vector's components in one coordinate system to its components in another. In order for a vector equation to hold in different coordinate systems, the equation must transform in the same way its vector quantities do.[18] Since vectors transform component-wise under coordinate changes, the vector equation must hold component-wise, each component equation transforming in just the way its vector quantities do. Remember those free-body diagrams from your high school physics class? That's what we were assuming in solving those problems: Newton's law can be resolved into its component equations, one for each dimension of the space. We were assuming the law holds component-wise.

Second, Newton's equation assumes that any coordinates we use to describe a system will be of a particular kind. The form of the equation above (equation 1) presupposes we can describe systems' motions in a space that admits of global Cartesian coordinates. As a result, the equation will take a different form when describing motions for which natural coordinates are not rectangular.[19]

To see this, think of different coordinate systems for the plane in terms of their (orthonormal) basis vectors. In a Cartesian coordinate system, the basis vectors are constants: they don't change with position; see Figure 2. The unit vectors ($\vec{e}_1$ and $\vec{e}_2$ in the figure) remain oriented in the same way independent of location in the plane. In a polar coordinate system, on the other hand, the basis vectors ($\vec{e}_\theta$ and $\vec{e}_r$ in the figure) do depend on position. (They are nonuniform vector fields.) In polar coordinates, we define *local* basis vectors: they are mutually orthogonal at each point, but

[18]A vector equation is equivalent to an equation between components: $\mathbf{u} = \alpha\mathbf{v} + \beta\mathbf{w}$ is true iff $u_i = \alpha v_i + \beta w_i$ is true for all $i = 1, \ldots, n$ (Johns, 2005, 576).

[19]We can express these things in coordinate independent terms. For now, picturing the difference in allowable coordinates gives an intuitive picture of the difference with Lagrangian mechanics, which we'll be coming to shortly.

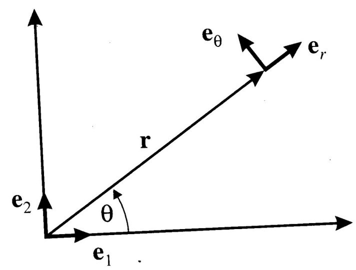

Figure 2: Polar coordinates on the plane; image from Greiner (2004, 247)

their orientation changes with position.[20] The polar basis vectors vary with position in the plane, unlike the Cartesian basis vectors.[21]

It turns out that for some classical systems, natural coordinates are not the Cartesian ones. The pendulum is an example of this. The direction of the force on the pendulum bob is always downward; the total force remains parallel to $\vec{e}_2$. The direction of the acceleration, however, does not remain constant: it changes with location, here given by angle θ. This means that the component of the force that's in the direction of the acceleration will also change with θ, which itself changes with time.[22] The total force has the same direction everywhere along the particle's path, while the tangent to the path (the velocity), and with it, its first time derivative (the acceleration), is continually changing. In Cartesian

[20]See Greiner (2004, ch. 10), Johns (2005, 501).

[21]The reason one set of basis vectors is variable and the other is not is that the transformations between Cartesian and polar coordinates don't only involve linear combinations of the original coordinates. We can see this by looking at the equations relating the components in one system to those in the other: Cartesian to polar: $x = r\cos\theta$, $y = r\sin\theta$; polar to Cartesian: $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Or by looking at the relations between the basis vectors: $\vec{e}_r = \left(\frac{\partial\vec{r}}{\partial r}\right)_\theta = \vec{e}_x\cos\theta + \vec{e}_y\sin\theta$ and $\vec{e}_\theta = \left(\frac{\partial r}{\partial\theta}\right)_r = -\vec{e}_x\sin\theta + \vec{e}_y r\cos\theta$. Or by comparing area elements: in Cartesian coordinates, $dA = dxdy$; in polar coordinates, $dA = r\, drd\theta$; i.e., the Jacobian of the transformation matrix has determinant r (not 1, as would be the case if the area element were preserved).

[22]For simple harmonic motion along a straight line, the force is also a function of a position that varies with time; but the total force is always in the direction of the displacement, unlike for the pendulum. Assume a flat space with a well-defined notion of 'same direction as'. Parallel-transport the total force vector along the path, and it always points in the same direction. We'll return to this.

coordinates, the force is always in the direction of a coordinate axis. This is not the case for the acceleration. In polar coordinates, on the other hand, one of the coordinate axes (in the direction $\vec{e}_\theta$) remains parallel to both the acceleration and the force giving rise to that acceleration. The polar coordinate system "moves with" the pendulum. The Cartesian coordinate system does not.

The pendulum reveals something interesting about Newton's law. Recall that the law holds component-wise: it says that the net force *in the direction of the acceleration* gives rise to an acceleration. Newton's law relates the component of the total force in the direction of the acceleration, to the resulting acceleration, in that same direction. Yet during the pendulum's motion, the component of the force changes with position. Since this component changes with position, the component form of the equations change with position; which is just to say that the vector equation changes with position. Newton's law assumes that a given vector, such as the force on the pendulum, will have the same components at any location along the path. But this is a special property of Cartesian coordinate systems, for which the basis vectors themselves don't vary from location to location. Equation 1, in other words, assumes Cartesian coordinates, and will have to change mathematical form to accommodate a pendulum motion.

You might reply that the pendulum is accelerating, so that this is not an inertial frame, in which case Newton's law doesn't apply. Or you might point out that we can rewrite Newton's law in terms of angular quantities, such as angular velocity and momentum, and solve the problem that way (at least for the small-angle approximation near the bottom of the swing). Or, alternatively, that we can perform a change of variables and rewrite Newton's equation in polar coordinates, a version of the law that will characterize the pendulum's motion.

But that would be an unsatisfying end to the discussion. Although Newton's law, transformed to polar coordinates, will describe the pendulum's motion, it is puzzling that it should take a different mathematical form in those coordinates.[23] At the bottom of the swing, the pendulum

[23]By different mathematical form, I mean that we cannot simply take the equation $F = m\ddot{x}$ and replace x and its derivatives by θ and its derivatives. The reason is that the relation between the Cartesian and polar coordinates themselves is more complicated than that: see note 21.

approximates simple harmonic motion, and we don't think the law needs to be altered to describe simple harmonic motion along a straight line. (Recall Galileo's insight that simple harmonic motion is equivalent to uniform circular motion projected onto the diameter of the circle of motion.) More, we know that we can solve the problem using conservation of energy and the principles of Newtonian mechanics, without worrying about whether we're using rectangular or some other coordinates. In short, it is puzzling why Newton's law *should* prefer rectangular coordinate systems, when there are non-rectangular coordinate systems that are perfectly legitimate, indeed natural, for describing some systems. All of which suggests there ought to be a version of the law that applies regardless of whether a system is undergoing motion in a straight line or in a circle (whether uniform or simple harmonic). It suggests there should be a version of the law that is more coordinate-independent than equation 1. This, we'll see, is what the Lagrangian formulation gives us.

Enough foreshadowing—just enough, I hope, to get you intrigued. We'll return to these examples.

4. Lagrangian mechanics

Now for a quick review of Lagrangian mechanics. Once again we need two sets of coordinates to completely specify the state of a system at a time. But there's an important difference. Unlike the Newtonian equations, the Lagrangian equations are given in terms of *generalized coordinates*.[24] Generalized coordinates are any set of independent parameters that together completely specify the state of a system. The two sets of Lagrangian coordinates are the generalized positions, q_i, and their first time derivatives, the generalized velocities, $\dot{q}_i$ (i from 1 to n, for n particles).

As we saw above, in Newtonian mechanics we generally use ordinary position and momentum variables, given in regular Cartesian coordinates. The equations of motion still hold in other coordinate systems, but their mathematical form will change if we use, say, polar or any other non-rectangular, curvilinear coordinates.[25] That is what we saw for the

[24]The equations can be given without mentioning coordinates at all: $\mathbf{L}_\Delta \theta_L - dL = 0$. This is another important feature, though one we won't need for the basic point here.

[25]Curvilinear coordinate systems may have curved coordinate lines. They can have

pendulum, where we performed a change of variables and resolved the force on the system, a force which depends on an angular displacement, into its components in the chosen rectangular coordinate system.

The Lagrangian equations of motion, on the other hand, allow for *any* set of generalized coordinates and their first time derivatives. Different such coordinates can be used in the very same mathematical form of the equations.[26] In order to specify the state of a system of n particles freely moving in a three-dimensional space, we will need a total of $6n$ generalized coordinates: three generalized positions and three generalized velocities for each particle in the system.[27] But we needn't choose $6n$ rectangular coordinates, nor even $6n$ curvilinear coordinates. Any set of $6n$ independent parameters will do, so long as they completely specify the state. Thus, the generalized positions needn't resemble what we ordinarily take to be position coordinates. They needn't even have the dimensions of length. Depending on the problem, we might want dimensions of energy, or length squared, or a dimensionless quantity. For the pendulum, for example, we can use an angle as the generalized "position"; the generalized "velocity" is then the first time derivative of that angle. It turns out that we can treat this angle as if it were an ordinary rectangular coordinate, and everything goes through just as if we were using the more familiar coordinates. We'll see an example of this in a moment.

This difference in allowable coordinates yields a difference in the statespace. Both the Newtonian and Lagrangian statespaces will be $6n$-dimensional manifolds. But the Lagrangian statespace, called *configuration space*, is a bit different from the Newtonian one. Note that this is not the space of ordinary position configurations. That space, also called 'configuration space', is a $3n$-dimensional manifold, each point of which represents a set of particle positions in three-dimensional physical space. Each point in the Lagrangian statespace picks out *both* a generalized posi-

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variable unit vectors, as we saw above for polar plane coordinates, unlike Cartesian coordinate systems. The requirement is that the transformation between allowable coordinate systems be locally invertible at each point.

[26]For invariance of the equations under changes of coordinates, see e.g. Johns (2005, 2.9).

[27]If there are equations of constraint, not all $6n$ coordinates will be independent. For m constraint equations, there will be $6n - m$ independent coordinates.

tion and the corresponding generalized velocity for all the particles in a system. This statespace comprises a set of possible configurations (which needn't be given in ordinary position coordinates) plus a space in which we represent their first time derivatives. This is a *tangent bundle* of ordinary configuration space: the $3n$-dimensional (generalized-coordinate) configuration space, Q, together with the $3n$-dimensional tangent space, $T_q Q$, at each point q in Q. We need the tangent spaces in order to define the generalized velocities, which are tangent vectors: the generalized velocities lie along tangents to the possible trajectories in Q.[28] Unlike ordinary configuration space, then, the Lagrangian statespace is a $6n$-dimensional manifold, TQ, where each point picks out a generalized coordinate-generalized velocity pair for all the particles in the system.[29]

The Lagrangian equations of motion determine how the point representing the state of a system moves through this statespace over time. In Newtonian mechanics, the total force on a system determines its behavior. In Lagrangian mechanics, the motion is given by a scalar function, called the *Lagrangian*, L. At each point in the statespace, this function assigns a number, (typically[30]) equal to the kinetic minus potential energy of the system. Given a system's initial state and Lagrangian, the equations of motion determine a unique history.

For a system with n degrees of freedom, such as the three spatial dimensions we're assuming here, the Lagrangian formulation gives a set

[28]For more detail, see Abraham and Marsden (1980, 1.6); Schutz (1980); Arnol'd (1989, 81). Typically, TQ is constructed by attaching the tangent plane $T_q Q$ to each q in Q. I say "typically" since this view of the statespace as being built up from more fundamental sub-spaces Q and TQ is one I think we must ultimately reject, for reasons given below.

[29]Keep in mind that the base manifold Q needn't resemble ordinary three-dimensional space in any obvious way. For a single free particle, it may be isomorphic to physical space, but this is a special case. In general, the q_i needn't represent configurations in three-dimensional space in any obvious way.

[30]I ignore time-dependent Lagrangians, non-conservative forces, and non-holonomic constraints, though the theory can handle these too; indeed, more easily than can the Newtonian one. Some sources I found particularly useful: Abraham and Marsden (1980); Schutz (1980); Burke (1985); Arnol'd (1989); Marion and Thornton (1995); Arnol'd et al. (1997); José and Saletan (1998); Marsden and Ratiu (1999); Isham (2003); Nakahara (2003); Goldstein et al. (2004); Szekeres (2004); Johns (2005); Penrose (2005, ch. 20).

of n second-order differential equations, one for each particle in each coordinate direction (one for each degree of freedom):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.\text{[31]} \tag{6}$$

These equations uniquely determine a system's motion for an initial state given by $2n$ values, the initial n q_i's and n $\dot{q}_i$'s. A solution, found by integrating the equations of motion, has the form of a function, or trajectory, on Q, $q(t)$.

Return to our examples. A particle moving on a straight line is a system with one degree of freedom, for which the theory yields one second order equation. We need one generalized position coordinate to characterize the motion. A simple choice would be the distance measured along the line from a chosen origin; label this s. The generalized velocity is then the first time derivative of this coordinate. The equation of motion is as above, equation 6, with s as the generalized coordinate: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$.

Thus, if the particle undergoes simple harmonic motion, for which $F(s) = -ks$, the Lagrangian equals the kinetic minus potential energy of the system: $L = T - V = \frac{1}{2}m\dot{s}^2 - \frac{1}{2}ks^2$. So $\frac{\partial L}{\partial s} = -ks$, $\frac{\partial L}{\partial \dot{s}} = m\dot{s}$, and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = m\ddot{s}$. Plug into equation 6, and we get the same form for the equation of motion that we did using Newton's law (equation 3), with the Newtonian position coordinate x and its time derivatives replaced by the Lagrangian generalized position coordinate s and its time derivatives:

$$m\ddot{s} = -ks. \tag{7}$$

Plug in a particular initial state, given by initial values $s(t_0)$ and $\dot{s}(t_0)$, and integrate to get the solution (see note 13).

If the particle is in free fall, take the generalized position coordinate to be the vertical displacement away from the height at which the particle

[31] Expand time derivatives to see these as second order:

$$\frac{\partial^2 L}{\partial \dot{q}^i \dot{q}^j}\ddot{q}^j + \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\dot{q}^j + \frac{\partial^2 L}{\partial t \partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0.$$

begins to fall; call this h. Then $T = \frac{1}{2}m\dot{h}^2$ and $V = -mgh$, so that $L = T - V = \frac{1}{2}m\dot{h}^2 + mgh$. Plug into equation 6 and we get $m\frac{d\dot{h}}{dt} - mg = 0$, or

$$\ddot{h} = g. \tag{8}$$

Again, we get the same form for the equation that we did using Newton's law (equation 4), with the Newtonian position coordinate x and its derivatives replaced by the generalized coordinate h and its derivatives.

Pause for a moment to consider the statespace for a particle moving on a straight line (whether simple harmonic motion or free fall). The statespace for this system is a two-dimensional space. It is the tangent bundle of a one-dimensional line (or circle): see Figure 3. Note that the

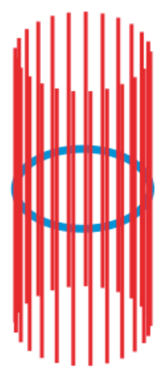

Figure 3: Illustration of a tangent bundle, the structure of the Lagrangian statespace for a particle constrained to one-dimensional motion; image by MIT OCW. The circle represents the base manifold Q, coordinatized by generalized position q. The lines (fibers), which represent the tangent spaces T_qQ above each q in Q, are coordinatized by the generalized velocity $\dot{q}$.

lines in the figure are all tangent to the base space, while at the same time, none of the lines cross; the figure represents the latter feature at the expense of the former. (This trivial bundle is one of the only easily visualizable tangent bundles.) The circle (the base manifold Q) represents the generalized position coordinate q (s or h, above). The lines (the fibers T_qQ) above each point in the circle represent the generalized velocity $\dot{q}$

($\dot{s}$ or $\dot{h}$ above).[32]

Each point $(q, \dot{q})$ in the statespace represents a possible state of the system, and different curves through the statespace represent different possible histories. For any particular value of q, there will be many possible trajectories that pass through that value—all the curves representing the particle with different generalized velocities at that point. When $\dot{q}$ is also given, the trajectory will be unique. Thus, each point in the statespace lies on, or determines, a unique trajectory. Different trajectories pass through at most one point on each fiber, corresponding to a unique history for an initial state. One nice feature of the tangent bundle statespace is that it separates out the possible trajectories in this way.

Importantly, the Lagrangian formulation yields the very same procedure for a particle traveling along an arclength of a circle. Consider the plane pendulum, only now forget about the rectangular coordinates we used in the last section: see Figure 4. We could start by using rectangular coordinates, calculate the Lagrangian in terms of those, and then use the transformation equations between coordinate systems to find the solution. But things are a lot simpler if we notice right away that the only generalized coordinate we'll need to characterize the motion is θ. L can be given entirely in terms of θ and $\dot{\theta}$: since $T = \frac{1}{2}m(l\dot{\theta})^2$ and $V = -mgl\cos\theta$, $L = T - V = \frac{1}{2}m(l\dot{\theta})^2 + mgl\cos\theta$. Calculate the derivatives $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$, $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$, and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$, and plug into equation 6. We find the same equation of motion that we did before using Newton's equation (equation 5), with the generalized coordinate θ and

[32] It is a bit more complicated than this. More is needed to ensure that the fibers over different on the base manifold are related to each other in the right way, the farther we go away from the initial circle. And even if two values along the circle are very close, we need more to ensure that their tangent lines must stay close throughout the lengths; see José and Saletan (1998, 94). That is, we haven't yet made the tangent bundle itself a differential manifold. The reason is that just joining the tangent planes to each point of the base manifold M doesn't create a differential manifold unless the planes are joined or "glued" together to in the right way. (In this case, so that the resulting TM is a cylinder in the usual sense). The gluing together is done by defining charts on TM to distinguish local neighborhoods telling us which points in one T_mM are close to others on a neighboring T_mM. We do this by using a local chart on M to construct a local chart on TM. This yields an atlas for TM, so that TM becomes a $2n$-dimensional manifold. See José and Saletan (1998, 101-102).

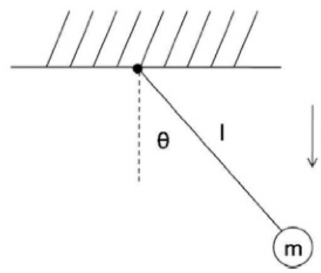

Figure 4: Pendulum; image by MIT OCW

its derivatives replacing the Newtonian coordinate x and its derivatives:

$$-g\sin\theta = l\ddot{\theta}. \tag{9}$$

Again the statespace looks like the one shown in Figure 3, where now the base manifold represents θ and the fibers represent $\dot{\theta}$.

Notice what we did not do for the pendulum. We did not write out equations for any of the forces on the system. We did not write out constraint equations arising from those forces. We did not write out any transformation equations between Cartesian and other kinds of coordinate systems. We could have done all these things. But we didn't have to do them in order to solve the problem using these equations.

According to the Lagrangian dynamics, it doesn't matter if the path of the particle is straight or curved.[33] The important feature is that the particle is constrained to one-dimensional motion, which signals the need for one generalized coordinate. We can choose, right off the bat, to use a coordinate that's natural for the system, and everything goes through just as if we were using ordinary rectangular coordinates. Just substitute s and $\dot{s}$, or h and $\dot{h}$, or θ and $\dot{\theta}$, for q and $\dot{q}$ in equation 6.

The end result—the solution representing the path of the particle through ordinary space, described by a curve through the relevant statespace[34]—is, in a certain sense, the same whether we use the Newtonian or

[33]Whether a path is curved can be characterized by the intrinsic structure of the space.

[34]The solution describes a curve through TQ, which is projected onto Q.

Lagrangian equations. Depending on the system in question, the calculations might be easier with one or the other. It was easier to solve for the pendulum motion using the Lagrangian equations, for instance, since we could calculate the potential and kinetic energies directly in terms of the coordinate θ and its derivatives, without having to calculate any forces or transform to other coordinate systems. But the set of possible histories we end up with is the same. For each of the systems above, we get the same equation of motion regardless of which formulation we start out with. The only difference is the variables appearing in that equation, a difference stemming from different ways of describing the initial set-up.

And so it seems as though any difference between Newtonian and Lagrangian mechanics can be no more than of a practical, ease-of-calculation kind. Each theory yields a set of second order differential equations, which we use to find the accelerations at the initial time; integrate for the accelerations at a later time; then treat the later time as the initial one, iterating to "unroll" the trajectory on Q. This procedure is the same with either formulation, yielding the same set of possible histories for a given system: the same set of paths through ordinary space, described by curves in the theories' respective statespaces.

In this sense, and as any classical mechanics textbook will demonstrate for you, the Newtonian and Lagrangian equations of motion are interderivable. In this sense, and as any classical mechanics book will say to you, the two formulations are *equivalent*. Newtonian mechanics and Lagrangian mechanics are mere notational variants. Just a difference in the notation being used, not a difference in the world being described.

5. Cross-structural comparisons

Although Lagrangian and Newtonian mechanics are equivalent in the above sense, this is not the only sense of equivalence available. In general, we have to be careful when claiming an equivalence, or similarity, or isomorphism, between two things, whether concrete objects, scientific theories, abstract spaces, what have you. What kind of equivalence holds, and whether it holds, will depend on the sense of similarity being used.

Remember that the way we compare the structures of spaces in physics is adapted from the way we compare the structures of spaces in mathematics. In mathematics, given a space and the set of allowable coordinates on it, we can find the structure of the space from the invariant quantities under changes in coordinates. (In a Euclidean plane, for example, the distance measure is invariant under translations and rotations of coordinate axes.) In coordinate-independent terms, the structure of a space is given by its automorphism group, the group of functions from the space onto itself that preserve its structure. For two different spaces, we can compare their structures by looking for a similarity mapping, or isomorphism, between them; that is, a bijective mapping between the underlying sets, for which the map and its inverse are both structure-preserving.[35]

Comparisons of structure in mathematics, then, are given by mappings between spaces (where this can be from a space onto itself). Two different spaces are deemed equivalent when there is a relevant structure-preserving mapping between them. Importantly, not when there is an isomorphism, full stop; but an isomorphism *relative to* the structure being preserved.[36] Think of these spaces as being built up, layer by layer, from a basic set of points, each layer with its own mathematical structure and an associated mapping that preserves that structure.[37] In order for two "bare" sets to be isomorphic, it suffices that they have the same number of elements; then there will be a bijection between them, a map that preserves their structure as sets.[38] For two groups to be structurally equivalent, there are further requirements: the isomorphism must put their elements in one-one correspondence in a way that preserves the identity and group operation.[39] For two topological spaces, the isomorphism must be a continuous transformation; two spaces are topologically equivalent when they can be deformed into each other by means of a continuous invertible mapping.[40] For differential manifolds, the isomorphism must preserve

[35]Isham (2003, 51).

[36]This is emphasized in Schutz (1980); Geroch (1985); Isham (1989, 2003).

[37]Cf. Burke (1985, 37); see Sklar (1974, ch. II), Szekeres (2004).

[38]This establishes a unique correspondence between their elements, so that, "from a purely set theoretic viewpoint, these can be regarded as the 'same' set. Bijections will exist between X and Y iff they have the same 'number' of elements" (Isham, 1989, 2).

[39]This mapping, which for groups "preserves, in the strongest sense, all the structure available" (Geroch, 1985, 17), is called a homomorphism. For two groups, then, "'isomorphic'...just means identical in their group properties" (Schutz, 1980, 13).

[40]This mapping is called a homeomorphism. Intuitively, it maps points close together

their differential structure; the map and its inverse must be differentiable (to the same degree).[41]

You get the idea. If there is a bijection between two spaces, we can say that they are "the same" or "equivalent" with respect to some structure. But they may still differ in further structure. There's a metric-preserving map between a Euclidean plane with a preferred location and a Euclidean plane without one. Yet the plane with the preferred location has some structure the other one lacks. This extra structure is not invariant under space translations, whereas the entirety of the other plane's structure is.[42] In physics, too, we must be careful to keep track of the structure being preserved when comparing two spaces. The Minkowski spacetime of special relativity and the Galilean spacetime of classical mechanics share some structure: they are both flat spaces (with zero intrinsic curvature) of the same dimensionality. But the classical spacetime has more structure: it has a notion of absolute simultaneity which the Minkowski spacetime lacks. The classical spacetime isn't invariant under transformations that alter the simultaneity planes; the special relativistic spacetime is.[43]

Keep this in mind as we turn to Newtonian and Lagrangian mechanics. A comparison of structure will reveal an important way in which these theories are not equivalent: They differ in their statespace structure.

Remember our rule for inferring structure to the world. In the case of spacetime, our rule says: infer the least amount of spacetime structure presupposed by the fundamental dynamical laws. One mark of a lesser structure is a symmetry, or invariance, in the laws. Therefore, on the basis

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in one space to points close together in the other, and points not close together to points not close together, preserving the topological structure. See Isham (2003, ch. 1).

[41] This mapping is called a diffeomorphism (Bishop and Goldberg, 1980, 37). A homeomorphism is thus a diffeomorphism without the differentiability requirement (Schutz, 1980, 30, 40).

Here is a helpful summing up from Isham: "A very important general idea in mathematics is that of a structure-preserving map between two sets that are equipped with the same type of mathematical structure. For example, in group theory this would be a homomorphism; in topology, a structure-preserving map is a continuous map between two topological spaces. In differential geometry, the role of a structure-preserving map is played by a 'C^r-function' between two manifolds" (2003, 68-69).

[42] The symmetric plane lacks a preferred point for an origin. In this sense, it is an affine space, a linear space minus the origin: Burke (1985, I.1).

[43] See Sklar (1974); Earman (1989); Maudlin (1994, ch. 2).

of invariances in the laws, our rule says, infer corresponding symmetries in the world.

Follow this rule. In special relativity, choose an inertial frame with a different velocity, and the laws remain the same. This is then a symmetry of the theory, and of the world the theory describes: there is no preferred velocity on this theory. As far as special relativity is concerned, there is no difference in the *world* corresponding to different choices of velocities. There is only a difference in equally allowable *descriptions* of that world.

In classical mechanics, choose a coordinate system with a different origin, and the laws remain the same. This is then a symmetry of the theory, and of the world the theory describes: there is no preferred location on this theory. As far as classical mechanics is concerned, there is no difference in the *world* corresponding to different choices of origin. There is only a difference in equally allowable *descriptions* of that world.

In Lagrangian mechanics, choose a different set of generalized coordinates, and the laws remain the same. This is then a symmetry of the theory, and of the world the theory describes: there is no preferred type of coordinate system on this theory. As far as Lagrangian mechanics is concerned, there is no difference in the *world* corresponding to different choices of generalized coordinates. There is only a difference in equally allowable *descriptions* of that world.

But in Newtonian mechanics, choose a different set of generalized coordinates, and the laws might *not* remain the same. This is *not* a symmetry of the theory, nor of the world the theory describes: there *is* a preferred type of coordinate system on this theory, namely, the linear, rectangular ones. As far as Newtonian mechanics is concerned, there is a difference in the world corresponding to different choices of coordinate system, not just a difference in equally allowable descriptions of that world.

This last sentence (if not entire paragraph) may be jarring. But note the parallelism with the paragraphs that came before. According to the general rule, the Lagrangian equations' invariance under coordinate changes is important. It indicates that no type of coordinate system is preferred on this theory, in the same way that no location is preferred on a theory invariant under space translations, and no rest frame is preferred on a theory invariant under Galilean or Lorentz transformations. Different choices of coordinate system simply don't *matter* to the theory; no one choice is more "correct" than any other. As in the case of spacetime, this suggests that the structure of a world governed by the theory is likewise symmetric. It suggests that the *world itself* doesn't care which kind of coordinates we use. The symmetric structure of the world is the reason the theory doesn't privilege any coordinate systems.

This is not the case in Newtonian mechanics. The equations of motion do change if we switch from Cartesian to polar coordinates, for example.[44] Some coordinates are "more correct" than others: the ones that preserve the simplest form of the equations. As in the case of spacetime, this suggests that the structure of a world governed by the theory is likewise *a*symmetric. It suggests that the *world itself* cares which coordinates we use. The asymmetric structure of the world is the reason the theory privileges some coordinate systems over others.

This difference shouldn't be entirely surprising. In Lagrangian mechanics, a scalar function determines the motion of a system, and scalars are invariant to coordinate transformations. (They can be defined as quantities that are so invariant.) More, the relationship between the dynamical quantities of the Lagrangian equations, defined via the tangent, is independent of coordinate system. In Newtonian mechanics, on the other hand, a vector function determines the motion. Unlike the Lagrangian scalar function, the "value" of the Newtonian vector function—the component of the force in the direction of the acceleration—is not the same regardless of the type of coordinate system.[45]

(For a system acted on by only conservative forces, such as the pendulum, we could instead solve the problem using conservation of energy, sidestepping calculations of component forces. In Newtonian mechanics, this greatly simplifies the problem. Viewed as a system acted on by the force of gravity, the pendulum undergoes two-dimensional motion; viewed as system adhering to conservation of energy, it effectively undergoes one-dimensional motion. But Lagrangian mechanics allows us to treat this as a one-dimensional motion from the start.)

[44] In the special case that we use Cartesian coordinates as generalized coordinates, the Newtonian and Lagrangian equations will be the same.

[45] Similarly, consider the variational calculus derivation of the Lagrangian equations from a least action principle. The condition for a curve to be an extremal of a functional does not depend on the choice of coordinate system: Arnol'd (1989, 59), (1997, 249).

The difference in allowable coordinates is thus more important than it may have first seemed. A difference in allowable coordinates means a difference in the laws' invariances. According to our rule, this means a difference in the structure of the world they describe. Since our rule says to infer the *least* structure indicated by the laws' invariances, it says that Newtonian mechanics presupposes the structure to pick out a preferred type of coordinate system; Lagrangian mechanics does not.

Note, this doesn't mean there is *no* absolute structure in a world governed by Lagrangian mechanics. The equations presuppose a certain amount of structure. The statespace must be such that we can pick out any point by a set of real numbers (it must have a manifold structure).[46] And it must be possible to describe systems' dynamical evolution with differential equations defined on that space (it must have a differentiable manifold structure). There may be further constraints besides (as we'll see in a moment). Yet when all is said and done, this is going to be less structure than the structure presupposed by Newtonian mechanics.

Hold that thought while we compare the theories' statespaces. So far we've been talking about the difference in allowable coordinates. That difference can be characterized directly by means of the theories' respective statespace structures.

The Lagrangian statespace is a tangent bundle, TQ, a particular type of vector fiber bundle. Locally, this space looks like a product space (of Q and T_qQ, for each q in Q).[47] Globally, though, the topology can be different from that of a vector product space; it can be 'nontrivial' on the whole. (A simple nontrivial vector bundle is a line bundle of a circle, twisted into a Möbius strip.[48]) Thus, global properties of the statespace are important. Even the statespace for our pendulum needs

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[46] A set of points M is a manifold if each point of M has an open neighborhood which has a continuous $1-1$ map onto an open set of $\mathbb{R}^n$. That is, M is locally 'like' $\mathbb{R}^n$; locally it has the same topology as that of $\mathbb{R}^n$. See Schutz (1980, 23-30); Isham (2003, 59-70).

[47] If M is an m-dimensional manifold an N is an n-dimensional manifold, the Cartesian product $M \times N$ has the structure of an $(m+n)$-dimensional manifold.

[48] See Schutz (1980, 2.9-2.11); Burke (1985, II.13); Baez and Muniain (1994, ch. 2); Sternberg (1994, 98); Isham (2003, ch. 5); Nakahara (2003, 171-2, 350-355); Penrose (2005, ch. 15). For detailed discussion of the intrinsic geometric structure of the tangent bundle, in particular for Lagrangian mechanics, see Crampin (1983); Morandi et al. (1990).

some global structure to ensure that the fibers join together in a smooth, non-overlapping way, forming an ordinary, 'untwisted' cylinder. For most systems, the statespace can't even be given a trivial structure; think of the statespace for a single particle constrained to move on the surface of a sphere (the tangent bundle of the two-sphere). Not only must the base space and the fibers be $3n$-dimensional spaces, but there must be the right kind of glue, or 'connection', joining the fibers into a unified whole.[49]

The Newtonian statespace can also be seen as a fiber bundle, but of a particular kind. The Newtonian equations assume that a given motion can be completely described using Cartesian coordinates. The structure of the statespace—the space on which these equations are defined—must reflect this. Thus, not only is this space a $6n$-dimensional manifold, but it is a manifold that admits of a global Cartesian coordinate system. Thinking of it as a vector bundle, this space is the trivial bundle, a vector product space for which, in particular, both the base space and each fiber are $3n$-dimensional Euclidean spaces. The reason we don't ordinarily think of it this way is that the base space, and corresponding tangent space, is flat; so that the position, velocity, and force vectors can all be thought of as "living in" the same space.[50] Since the base space is a flat $3n$ dimensional Eucildean space—the kind of space, that is, on which we can lay down global Cartesian coordinates[51]—and the tangent spaces are also $3n$-dimensional Euclidean planes, the entire statespace can be seen as (having the topology of) a $6n$-dimensional flat Euclidean space.[52]

[49]That is, it's not sufficient to say what the base and fiber of a bundle are, for there may be more than one way to construct such a bundle; we also need to specify the connection, which says how the fibers over nearby points in the base space are glued together. See Schutz (1980, 2.11); Penrose (2005, ch. 15); Maudlin (2007).

[50]In a flat space, the tangent spaces can be identified with one another (Schutz, 1980, 214).

[51]In other words: the kind of space on which one can lay down coordinates x, y, z such that the distance between two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

[52]Another way of putting this is to say that the connection on the bundle is flat. As Arntzenius puts it for the case of a flat spacetime: "In hindsight, it is natural to suppose that even in flat spacetimes, the structure of spacetime includes a 'connection', which determines what it is to parallel transport a vector along a path. It just happens that in flat spacetimes the connection is such that when one parallel transports vector one point to another, always end up with same vector regardless of path. Indeed, one can

The Lagrangian statespace also has a natural metric associated with it, given by the invariant quantity of the equations of motion.[53] But this is a Riemannian metric, a generalization of the Euclidean metric applicable to curved spaces and curvilinear coordinates. This geometric object allows us to calculate distances between nearby points, and to add those up to find the length of a path; it also gives the geodesics of the space. But the statespace needn't be flat for that; and, in general, it won't be.[54] Just think of the statespace of the pendulum.

Thus, the Lagrangian and Newtonian statespaces will share some amount of structure. Both are $6n$-dimensional differential manifolds, with the same differential structure. They are both (or can both be seen as) fiber bundles, with base space and fiber each of dimension $3n$. They both have the local topology of a vector product space. Locally, they look the same. Yet just as with the line bundle vs. the one twisted into Möbius strip, the two statespaces differ globally. Although the Lagrangian statespace has a definite structure, some of which is shared by the Newtonian statespace, the Lagrangian equations assume a general structure of which the Newtonian statespace is just a special kind. The flat connection and Euclidean metric of the Newtonian statespace is a *special case* of the curved connection and Riemannian metric of the Lagrangian statespace. This is reflected in the relative freedom of coordinates: in Newtonian mechanics, the vector quantities must be tangent to the possible particle trajectories in a Euclidean 3-space; in Lagrangian mechanics, the trajectories can be

take this as a definition of what it is for spacetime to be flat. The benefit of taking this as a definition of curvature, rather than defining flatness in terms of a metric, is that it can be applied to fiber bundles that have no metric structure" (ms, ch. 2).

[53]Although fiber bundles have a connection, they need not have a metric. The invariant of the Lagrangian equations is what endows the space with a Riemannian metric. We can also see this from the variational calculus formulation of the theory: the curve representing a system's history follows the geodesics of the statespace. (Again, I'm assuming conservative systems for which the Lagrangian is regular and can be written in the form of a homogeneous function of the $\dot{q}^2$'s.) Thus Szekeres (2004, 469) defines a Lagrangian system as an n-dimensional Riemannian manifold (M, g), together with a function $L : TM \to \mathbb{R}$, the Lagrangian of the system.

[54]Hence the trouble with the transformation from polar to Cartesian coordinates on the plane; see note 21. There will be a smooth coordinate transformation that which looks locally like a linear transformation; but the transformation itself is position dependent. See Burke (1985).

tangent to particle "trajectories" which don't in any obvious way trace out the path of the particle through ordinary space.[55]

That's why, when using Lagrangian mechanics to solve physics problems, we can choose generalized coordinates and a statespace geometry to match the motion of the system. Whereas Newtonian mechanics requires that we find the constraint forces keeping the pendulum bob on a curved path, for example, in Lagrangian mechanics we can build this into the structure of the statespace, then use coordinates that are natural for that space.

(Although above I said that both the Lagrangian and Newtonian theories yield second order equations of motion, the Lagrangian equations are only second-order if we think of them as defined on Q. Since the Lagrangian function is a function on the whole of TQ, one might think Lagrangian the equations are more properly seen as first-order equations on TQ.[56] Hence the reason that the structure of TQ is so useful: it separates the trajectories (no two can cross) and is a reflection of the first-order nature of the equations on TQ.)

Yet for all its flexibility, the Lagrangian formulation suffices to describe these systems' motions. This suggests that it, unlike Newtonian mechanics, somehow gets right to the heart of a given set-up, to the core features that are relevant to the dynamics. The Lagrangian formalism abstracts away from the features that don't, in the end, *matter* to the dy-

[55] See Johns (2005, 2.4). Usually the generalized coordinates, q_i, unlike Cartesian coordinates, won't even obviously divide up into convenient groups of three that can be associated together to form vectors. The requirement is simply that the coordinate transformations must be invertible in an open neighborhood of every point of configuration space, i.e., with nonzero Jacobian determinant.

[56] We can transform the equations to a set of first order equations on TQ of the form $\frac{dq^\alpha}{dt} = \dot{q}^\alpha$, $\frac{d\dot{q}^\alpha}{dt} = W^\alpha(q, \dot{q}, t)$. Combine the q^α and $\dot{q}^\alpha$ into a single set of $2n$ variables ξ^j, and we get the equations of motion on TQ as first order differential equations of the form: $\frac{d\xi^j}{dt} = f^j(\xi)$. The first n of these equations read $dq^\alpha/dt = \dot{q}^\alpha$ and the last are Lagrange's equations above (equation 6). See José and Saletan (1998, 93-97). (This is also apparent from the exterior differential form of the equations in note 24.) In other words, on TQ, the solution in generalized coordinates are the $2n$ functions $q_i(t)$ and $\dot{q}_i(t)$, and we interpret the $\ddot{q}_j$'s appearing in the equations as the first derivatives of the $\dot{q}_j$'s with respect to time. Note that the equations define a vector in the tangent space at each $\xi \in TQ$; but since TQ is itself a tangent bundle, these equations really involve the tangent bundle of a tangent bundle, $T(TQ)$, or T^2Q (José and Saletan, 1998, 103-104).

namics. It suggests that the proper statespace structure is not Euclidean, but Riemannian.

And as for these various systems and their statespaces, so too for the world and its statespace. The structure of the statespace of the world is a more general, looser structure in Lagrangian mechanics. We can again see this in two ways. The coordinate-dependent way: the Lagrangian equations are invariant under a wider range of coordinate transformations. The coordinate-independent way: the Lagrangian statespace has less structure. The Newtonian statespace just is the Lagrangian statespace, with an additional level of structure, picking out a privileged metric tensor as preferred, and correspondingly privileged coordinate systems.

But note what this suggests. According to Lagrangian mechanics, individual physical systems are distinguished by means of very general geometric and topological properties. These properties are built into a system's statespace, giving its intrinsic structure. The dynamical laws don't distinguish between systems that are the same in terms of these properties. In other words, in Lagrangian mechanics, dynamical systems are distinguished by means of their *statespace structure*.

This suggests that, as far as the Lagrangian dynamics is concerned, there is *no real difference* between a single free particle that is moving on a straight line and a particle moving on an arclength of a circle. Each system—and the plane pendulum, for that matter—has a Lagrangian statespace like the one depicted in Figure 3, with the same Lagrangian function defined on it. A system's statespace, remember, represents all its physically possible states, and all its physically possible histories for different initial states. According to this theory, then, there *is* no fundamental difference between the systems. There is nothing else in the fundamental physical state that can distinguish the two; not without positing additional structure to the world, beyond what's required of the dynamics. But our methodological rule tells us that we shouldn't posit any such additional structure. This means that there is no fact of the matter about whether the particle is traveling along a line or along an arc length of a circle: they are one and the same system, described in different ways.

As for Lagrangian mechanics and individual systems, so too for Lagrangian mechanics and the world. According to Lagrangian mechanics, there is no fact of the matter as to whether *our* world is properly seen as having a Euclidean distance measure or some other. Or rather: there is only a very general, local, neighborhood-wise distance measure, not one that holds globally. There is a fact of the matter as to the geodesics of the world, but that is all. Our rule tells us this is *all* there is to the fundamental structure of a classical world.

[NOTE: Here's the point at which I'm not sure what the conclusion is going to be. I still haven't figured out what I think. Two possibilities. I started out thinking my conclusion was: (1) LM and NM differ in statespace structure; we should take that structure seriously; hence they differ in what they say about the world; and since LM is less structure, it is the one we should use to make inferences about a fundamentally classical world. But now I'm wondering if this also commits me to saying all of the above, plus: (2) in LM, there is no real distinction between a world with one particle moving on an arc vs. a straight line, and more generally, that the fundamental structure of space in a classical world is not Euclidean. This is not to deny the reality of ordinary physical space, but to say that ordinary space gets its properties from the more fundamental statespace and its structure; and this means denying that ordinary space has the Euclidean distance structure we ordinarily assume it does.]

6. Conclusion

Lagrangian mechanics is the simpler, more coordinate-independent version of classical mechanics, requiring less structure. A follow-up to our rule says: when two theories, otherwise equivalent, differ in amount of structure presupposed, infer the theory with the lesser structure. This is an application of our rule, not just to infer the structure of the world according to a certain a theory, but to compare the structure of the world according to different, even allegedly equivalent, theories.

Since Lagrangian mechanics posits less structure, we should look to it, not Newtonian mechanics, to discover the fundamental nature of a classical mechanical world. But notice how little the structure this is. It lacks the distance structure of ordinary three-dimensional Euclidean space. It has a topological structure and differential structure and a general Riemannian metric. All of this is much less than the ordinary distance structure.

At this point, you might worry that the simplification to single-particle systems starts to matter. Interactions between particles occur via inter-particle forces, which act along the line between them. But the Lagrangian formulation, we know, does work for multi-particle systems. And its structure should allow for this: it still makes sense to talk of the shortest or straightest path between two particles even without assuming a Euclidean metric. What we ordinarily think of forces as acting along inter-particle distances are acting along the geodesics in the statespace. A Euclidean metric structure is not needed for dynamics.

Since Lagrangian mechanics gets by with less structure, we should look to it, not Newtonian mechanics, to infer the fundamental structure of a classical world. This yields an inference to a surprisingly spare fundamental structure: a structure so spare that it does not consider a particle moving on a straight line and a particle moving on an arclength of a circle to be genuinely different situations. They are not genuinely distinct possibilities for this single particle system. A surprising conclusion, perhaps, but one that is warranted by our general rule.

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