A Logically Coherent *Ante Rem* Structuralism

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Our Plan For Addressing the Issues

- Review an axiomatic theory of abstract objects.
- Show how mathematics is analyzed within the theory.
- Interpret the analysis as a form of structuralism.
- Revisit each issue in light of this theory, to show how the theory addresses it.
- As we tackle each issue, some technical details of the background theory (skipped in the review) are further elaborated.
- As we tackle each issue, examine how the present version of structuralism compares with other forms of structuralism.

Issues To Be Addressed by a Mathematical Structuralism

- In what precise sense is mathematics about abstract structure?
- In what precise sense are the elements of a structure incomplete?
- What are the essential properties of the elements of a structure?
- Can there be identity/distinctness between elements of different structures?
  - Is the natural number 2 the same as the number 2 of Peano Number Theory and are these identical to the number 2 of Real Number Theory?
  - Does the answer suffer from the Julius Caesar problem?
- Do the elements of a mathematical structure ontologically depend on the structure?
- Do the elements of a structure have haecceities?
- Is indiscernibility a problem for structuralism?

The Axiomatic Theory of Abstract Objects I

- Use a second-order, quantified S5 modal logic with:
  - 2 kinds of atomic formulas:
    $F^n x_1 \ldots x_n \ (n \geq 0)$ — ($x_1, \ldots, x_n$ *exemplify* F^n)
    xF^1 — ('x encodes F^1')
  - Distinguished predicate '$E!$' ('being concrete'),
  - λ-expressions: $[\lambda y_1 \ldots y_n\, \varphi]$ — (φ no encoding formulas)
  - Rigid definite descriptions: $\iota x\varphi$
- *Ordinary* ('$O!$') vs. *abstract* ('$A!$') objects:
  $$O!x =_{df} \Diamond E!x$$
  $$A!x =_{df} \neg\Diamond E!x$$
- Identity is defined:
  $$x = y =_{df} [O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)] \vee [A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)]$$
  $$F = G =_{df} \Box\forall x(xF \equiv xG)$$
  $$p = q =_{df} [\lambda y\, p] = [\lambda y\, q]$$
- β, η, and α conversion on λ-expressions, and the Russell axiom for definite descriptions.

The Axiomatic Theory of Abstract Objects II

- From β-conversion:
  $[\lambda x_1 \ldots x_n\, \varphi]y_1 \ldots y_n \equiv \varphi^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}$
  derive Second-order Comprehension:
  $\exists F^n \Box \forall x_1 \ldots \forall x_n (F^n x_1 \ldots x_n \equiv \varphi)$
  $\exists F \Box \forall x (Fx \equiv \varphi)$,
  $\exists p \Box (p \equiv \varphi)$
  where φ has no free Fs (or ps) and no encoding subformulas.
- Ordinary objects necessarily fail to encode properties:
  $O!x \rightarrow \Box\neg\exists F(xF)$
- Comprehension for Abstract Objects:
  $\exists x(A!x \,\&\, \forall F(xF \equiv \varphi))$, where φ has no free xs
  e.g., $\exists x(A!x \,\&\, \forall F(xF \equiv Fb))$
- Well-Defined Descriptions: $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$
- Proper Theorem Schema: $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \;\equiv\; \varphi^G_F$
  e.g., $\imath x(A!x \,\&\, \forall F(xF \equiv Fb))L \;\equiv\; Lb$

Analyzing Natural Mathematical Objects

- The Triangle.
  $\Phi_T =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv \Box\forall y(Ty \rightarrow Fy)))$
- The number of Gs.
  $\#G \;=_{df}\; \imath x(A!x \,\&\, \forall F(xF \equiv F \approx_E G))$
  Theorem: $\#F = \#G \equiv F \approx_E G$ (Hume's Principle)
- The extension of G.
  $\epsilon G =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv \forall y(Gy \equiv Fy))$
  Theorem: $\epsilon F = \epsilon G \equiv \forall x(Fx \equiv Gx)$ (Basic Law V)
- The direction of line a.
  $\vec{a} =_{df} \epsilon[\lambda x\, x\|a]$
  Theorem: $\vec{a} = \vec{b} \equiv a\|b$ (Directions)
- See Pelletier & Zalta 2000, Zalta 1999, Anderson & Zalta 2004.

Two Kinds of Mathematics

- Mathematics has an intuitive division: natural mathematics and theoretical mathematics.
- Natural mathematics: ordinary, pretheoretic claims we make about mathematical objects.
  The Triangle has 3 sides.
  The number of planets is eight.
  There are more individuals in the class of insects than in the class of humans.
  Lines a and b have the same direction.
  Figures a and b have the same shape.
- Theoretical mathematics: claims that occur in the context of some explicit or implicit (informal) mathematical theory, e.g., theorems.
  In ZF, the null set is an element of the unit set of the null set.
  In Real Number Theory, 2 is less than or equal to π.

Analyzing Mathematical Theories and Objects

- *p is true in T* ('$T \models p$') $=_{df} T[\lambda y\, p]$
  i.e., treat mathematical theories as objects that encode propositions
- **Importation**: For each formula φ that is an axiom or theorem of T, add to object theory the analytic truth $T \models \varphi^*$, where φ^* is the result of replacing every well-defined singular term κ in φ by κ_T. This validates the **Rule of Closure**:
  If $T \models p_1, \ldots, T \models p_n$ and $p_1, \ldots, p_n \vdash q$, then $T \models q$.
- **Reduction Axiom**: Theoretically identify (well-defined) individual κ_T as follows:
  - $\kappa_T = \imath x(A!x \,\&\, \forall F(xF \equiv\; T \models F\kappa_T))$
  - $0_{\mathrm{PNT}} = \imath x(A!x \,\&\, \forall F(xF \equiv\; \mathrm{PNT} \models F0_{\mathrm{PNT}}))$
  - $\emptyset_{\mathrm{ZF}} = \imath x(A!x \,\&\, \forall F(xF \equiv\; \mathrm{ZF} \models F\emptyset_{\mathrm{ZF}}))$
- **Consequence**: **Equivalence Theorem**:
  $\kappa_T F \equiv T \models F\kappa_T$

Analysis of Mathematical Language

- By mathematical practice (ignoring mathematical relations):
  $\vdash_{ZF} \emptyset \in \{\emptyset\}$ $\quad \vdash_{ZF} [\lambda x\, x \in \{\emptyset\}]\emptyset$
  $\vdash_{\mathfrak{R}} 2 \leq \pi$ $\quad \vdash_{\mathfrak{R}} [\lambda x\, x \leq \pi]2$
- By Importation, true readings of mathematical claims:
  $\text{ZF} \models \emptyset_{ZF} \in \{\emptyset_{ZF}\}$ $\quad \text{ZF} \models [\lambda x\, x \in \{\emptyset_{ZF}\}]\emptyset_{ZF}$
  $\mathfrak{R} \models 2_{\mathfrak{R}} \leq \pi_{\mathfrak{R}}$ $\quad \mathfrak{R} \models [\lambda x\, x \leq \pi_{\mathfrak{R}}]2_{\mathfrak{R}}$
- Consequences of the Equivalence Theorem:
  $\emptyset_{ZF}F \equiv \text{ZF} \models F\emptyset_{ZF}$ $\quad \emptyset_{ZF}[\lambda x\, x \in \{\emptyset_{ZF}\}]$
  $2_{\mathfrak{R}}F \equiv \mathfrak{R} \models F2_{\mathfrak{R}}$ $\quad 2_{\mathfrak{R}}[\lambda x\, x \leq \pi_{\mathfrak{R}}]$
- Since encoding is a mode of predication, unprefixed mathematical claims get true readings:
  - The null set *is* an element of the unit set of the null set.
  - 2 *is* less than or equal to π.

Mathematical Relations

- Use typed object theory, where i is the type for individuals, and $\langle t_1, \ldots, t_n \rangle$ is the type of relations among entities with types $t_1, \ldots, t_n$, respectively. Use Typed Comprehension:
  $\exists x^t(A^{\langle t \rangle}!x \;\&\; \forall F^{\langle t \rangle}(xF \equiv \varphi))$, where φ has no free x^ts
- Let R be a variable of type $\langle i, i \rangle$, $\boldsymbol{F}$ be a variable of type $\langle\langle i, i \rangle\rangle$, and $\boldsymbol{A!}$ denote the property of being abstract (type: $\langle\langle i, i \rangle\rangle$):
  $\exists R(\boldsymbol{A!}R \;\&\; \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \varphi))$, where φ has no free Rs
- **Importation**: For each axiom/theorem φ of T, add the analytic truths of the form $T \models \varphi^*$, where φ^* is the result of replacing every well-defined singular term κ and well-defined predicate Π in φ by κ_T and Π_T, respectively, thereby validating the **Rule of Closure**.
- **Reduction Axiom**: Theoretically identify relation Π:
  $\Pi_T = \iota R(\boldsymbol{A!}R \;\&\; \forall \boldsymbol{F}(R\boldsymbol{F} \equiv T \models \boldsymbol{F}\Pi_T))$
  $S_{\text{PNT}} = \iota R(\boldsymbol{A!}R \;\&\; \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \text{PNT} \models \boldsymbol{F}S_{\text{PNT}}))$
  $\in_{ZF} = \iota R(\boldsymbol{A!}R \;\&\; \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \text{ZF} \models \boldsymbol{F}\in_{ZF}))$
- Consequence: **Equivalence Theorem**:

$$\Pi_T\boldsymbol{F} \;\equiv\; T \models \boldsymbol{F}\Pi_T$$

Issue: How Is Mathematics About Abstract Structure?

- When we extend (the language of) object theory by importing the expressions and theorems of T into object theory, T's content becomes objectified. We may identify:
  The structure $T = \iota x(A!x \;\&\; \forall F(xF \equiv \exists p(T \models p \;\&\; F = [\lambda y\, p])))$
- Given what's expressed by $T \models p$, T is an an object that makes true all of the theorems of T!
- So structures are abstract objects.
- Moreover, we may define:
  x *is an element of* (structure) $T =_{df} T \models \forall y(y \neq_T x \rightarrow \exists F(Fx \;\&\; \neg Fy))$
  R *is a relation of* (structure) $T =_{df} T \models \forall S(S \neq_T R \rightarrow \exists \boldsymbol{F}(\boldsymbol{F}R \;\&\; \neg \boldsymbol{F}S))$
- So the relations and elements of a structure are abstract.
- Physical system K has the structure T if the relations of K exemplify the properties encoded by the relations of T or if there is an isomorphism between the relations and objects of K and T.

Issue: Are Structural Elements and Relations Incomplete?

- Answer: Yes, with respect to encoding, but not exemplification.
- Let x^t range over abstract entities of type t, and $F^{\langle t \rangle}$ range over properties of entities of type t, and $\bar{F}^{\langle t \rangle}$ denote $[\lambda y^t\, \neg F^{\langle t \rangle}y]$:

$$x^t \textit{ is incomplete } =_{df} \exists F^{\langle t \rangle}(\neg xF \;\&\; \neg x\bar{F})$$

- So the slogan "Mathematical objects possess only structural (relational) properties," has two readings (given the ambiguity of 'possess'): one is true and one false.
- Benacerraf's (1965) argument (from 'numbers have no properties other than structural ones' to 'the numbers aren't objects') fails for structural elements that both encode and exemplify properties.
- This undermines the counterexamples in Shapiro (2006) and Linnebo (2008) (e.g., 3 has the property of being the number of my children, being my favorite number, being abstract, etc.). Full retreat from incompleteness isn't justified.

Issue: Which Properties Are Essential To Structural Elements?

- Answer: their encoded properties.
- Reason: Because these are the properties by which they are theoretically identified via the Reduction Axiom and by which they are individuated by the definition of identity for abstract objects; they make them the objects that they are.
- Encoded mathematical properties are even more important than properties necessarily exemplified (e.g., not being a building, being abstract, etc.)
- Theory explains the asymmetry between Socrates and {Socrates}. Let M be modal set theory plus urelements. After importation:
  $\mathrm{M} \models s \in \{s\}_{\mathrm{M}} \quad \mathrm{M} \models [\lambda z\, s \in z]\{s\}_{\mathrm{M}} \quad \mathrm{M} \models [\lambda z\, z \in \{s\}_{\mathrm{M}}]s$
- The middle claim implies $\{s\}_{\mathrm{M}}[\lambda z\, s \in z]$, which in turn implies that {Socrates} essentially has Socrates as an element; but we can't abstract out any properties about Socrates from these claims: they involve encoding claims.

Issue: Do Elements Ontologically Depend on Their Structure?

- Answer: A structure and its relations/elements ontologically depend on each other.
- Reason: The structure and its relations/elements all exist as abstractions grounded in facts of the form $T \models p$.
- This applies to both algebraic and non-algebraic mathematical theories.
- Our answer therefore is in conflict with Linnebo 2006: "a set depends on its elements in a way in which the elements don't depend on the set" (72); "the identity of a singleton doesn't depend on any other objects or on the hierarchy of sets" (73); "we can give an exhaustive account of the identity of the empty set and its singleton without even mentioning infinite sets" (73); "no set is strongly dependent on the structure of the entire universe of sets. For every set can be individuated without preceeding via this structure" (79).

Issue: Can There Be Cross-structural Identity/Distinctness?

- The theory yields: $2 \neq 2_{\mathrm{PNT}} \neq 2_{\mathfrak{R}}$.
  (They encode different properties. Cf. Frege, Shapiro 2006, 128.)
- The theory yields: $2 \neq \{\{\emptyset\}\}_{\mathrm{ZF}} \neq \{\emptyset, \{\emptyset\}\}_{\mathrm{ZF}}$.
  (These encode different properties.)
- No Caesar problem: $2_T \neq$ Caesar, given that Caesar is ordinary.
- When T and T' have the same theorems (e.g., because T' has a redundant axiom), or are notational variants, we collapse them (and their objects) prior to importation.
- What can we say about the structure PNT and the ZF set ω? Answer: There is an isomorphism between the structural elements of PNT and the members of ω_{ZF}, just as there is an isomorphism between the elements of ω_{ZF} and ω_{ZFC}.

A Comparison: Linnebo (2006) and the Present Theory

- Linnebo relies on Fine's notion of ontological dependence, and Fine's theory of essence. Our theory of essence, and explanation of the asymmetry between s and $\{s\}$, is different.
- Linnebo uses (a) abstracts for isomorphism types and (b) a Fregean biconditional for offices. Both are problematic.
- Linnebo's asymmetry in pure set theory gets it purchase from focusing on unit set theory: Null Set, Extensionality, and Unit Set Axiom, i.e., $\forall x \exists y \forall z (z \in y \equiv z = x)$. If you restrict your attention to unit set theory, no reference to large cardinals, for example, is made when individuating unit sets.
- But the individuation of any set in unit set theory goes by way of Extensionality, which quantifies over every set. Thus, the individuation of both $\emptyset$ and $\{\emptyset\}$ (or indeed, of any unit set) is defined by the whole structure of unit set theory.
- Similarly, in full ZF, the individuation of both $\emptyset$ and $\{\emptyset\}$ (or indeed, any set) quantifies the whole structure of ZF-sets.

Issue: Do the Elements of a Structure Have Haecceities?

- Answer: No.
- Reason: Identity (simpliciter) is defined in terms of encoding formulas. Neither '$[\lambda xy\, x{=}y]$' nor '$[\lambda x\, x{=}a]$' are well-defined.
- Background: abstract objects can be modeled by sets of properties. But you can't, for each distinct set b of properties, have a distinct property $[\lambda x x{=}b]$ (violation of Cantor's Theorem).
- You can define $=_E$ as $[\lambda xy\, \forall F(Fx \equiv Fy)]$, and this is well-defined and denotes a relation (by Comprehension), but it is well-behaved only with respect to ordinary objects.
- The following is a theorem of object theory (a, b are abstract objects): $\forall R \exists a, b(a \neq b \;\&\; [\lambda x\, Rxa] = [\lambda x\, Rxb])$
- By letting R be $=_E$, we get: $\exists a, b(a \neq b \;\&\; \forall F(Fa \equiv Fb))$
- So there are distinct abstract objects (i.e., they encode different properties) that are indiscernible from the point of view of exemplification. (Picture of Aczel models.)

Another Example: The Case of i and $-i$

- To get $\mathbb{C}$, we take the axioms for $\Re$ and add the following:
  $i^2 = -1$
  (Strictly speaking, $i^2 =_{\mathbb{C}} -1$.)
- Objection: The structural elements i and $-i$ are collapsed in (our) structuralism.
- Reason: Any formula $\varphi(x)$ with only x free in the language of complex analysis that holds of i also holds of $-i$, and vice versa. Thus, i and $-i$ are indiscernible and after importing $\mathbb{C}$ we have $\mathbb{C}\models Fi \equiv \mathbb{C}\models F{-i}$. One might try to argue, by the Equivalence Theorem, that $iF \equiv \mathbb{C}\models Fi$ and $-iF \equiv \mathbb{C}\models F{-i}$. It would then follow that $iF \equiv -iF$. Therefore, $i = -i$, by the definition of identity for abstract objects.
- Response: This argument is blocked because i and $-i$ are not elements of the structure $\mathbb{C}$.

Issue: Does Indiscernibility Pose a Problem for Structuralism?

- Answer: No. Consider dense, linear orderings, no endpoints:

Transitive:	$\forall x, y, z(x < y \;\&\; y < z \rightarrow x < z)$	Irreflexive:	$\forall x(x \not< x)$
Connected:	$\forall x, y(x \neq y \rightarrow (x < y \vee y < x))$	Dense:	$\forall x, y \exists z(x < z < y)$
No Endpoints:	$\forall x \exists y \exists z(z < x < y)$		

- Aren't all the elements of this structure ('D') indiscernible?
- Answer: No. Reason: There are no elements of D. D is defined solely by general properties of the ordering relation $<_D$, which encodes such properties of relations as: $[\lambda R\, \forall x \neg xRx]$, $[\lambda R\, \forall x, y, z(xRy \;\&\; yRz \rightarrow xRz)]$, etc.
- Analogy: A novel asserts, "General X advanced upon Moscow with an army of 100,000 men". There aren't 100,002 characters, but only 3 (General X, Moscow, and the army of 100,000 men).
- Distinguish the model-theoretic notion 'object of theory T' (i.e., a value of a bound variable) and the object-theoretic notion 'element of structure T'. We can't appeal to the former in a philosophy of mathematics, since the former assumes the very mathematical language we're trying to analyze.

Formal Solution

- Our procedure: import φ of T into object theory by adding $T \models \varphi^*$, where φ^* is the result of replacing all the well-defined singular terms κ in φ by κ_T.
- *x is an element of* (structure) $\mathbb{C} =_{df} \mathbb{C} \models \forall y(y {\neq_{\mathbb{C}}} x \rightarrow \exists F(Fx \;\&\; \neg Fy))$
- By this definition, i and $-i$ aren't elements of $\mathbb{C}$.
- Our procedure for interpreting the language of $\mathbb{C}$: *before importation*, replace every theorem of the form $\varphi(\ldots i \ldots)$, by a theorem of the form: $\exists x(x^2 + 1 = 0 \;\&\; \varphi(\ldots x \ldots))$, and then import the result.
- Under this analysis, i and $-i$ disappear and we are left with structural properties of complex addition and complex multiplication. E.g., for complex addition $+_{\mathbb{C}}$, for each theorem $\exists x(x^2 + 1 = 0 \;\&\; \varphi(\ldots x \ldots))$, we can abstract out properties encoded by $+_{\mathbb{C}}$ of the form $[\lambda R\, \exists x(x^2 R 1 = 0 \;\&\; \varphi(\ldots x \ldots))]$. Similar techniques can be used for complex multiplication $\times_{\mathbb{C}}$.

Elements and Symmetries

- What gives a mathematical structure its *structure*? It is the relations of the theory. Without relations, there's no structure.
- An *element* of a structure must be uniquely characterizable in terms of the relations of the structure—it must be discernible.
- Indiscernibles arise from symmetries (non-trivial automorphisms) of the structure. Mathematicians working *with* a structure find it useful to give separate names to indiscernibles. But these names don't denote elements of the structure. After all, the names are arbitrary and there is nothing (i.e., no property) within the theory that distinguishes the indiscernibles from each other.
- The mathematician's use of 'i' and '$-i$' in $\mathbb{C}$ is different than their use of '$1_{\mathbb{C}}$' and '$-1_{\mathbb{C}}$'. The naming of 1 and -1 is not arbitrary — you can't permute $1_{\mathbb{C}}$ and $-1_{\mathbb{C}}$ and retain the same structure. So it makes sense to say that 'i' and '$-i$' do not denote objects the way that '$1_{\mathbb{C}}$' and '$-1_{\mathbb{C}}$' do.